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ORDM 2-1

ORDNANCE CORPS MANUAL

DESIGN OF FIRE CONTROL OPTICS

VOLUME 1

Application of Fundamental Laws of Geometrical Optics to Individual Optical Components

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Volume I

Application of Fundamental Laws of Geometrical Optics to Individual Optical Components

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SECTION I

FUNDAMENTAL LAWS OF GEOMETRICAL OPTICS

1. General

At a surface separating two media the direction of the ray of light will be changed abruptly. One portion of the light will be turned back into the first medium and we say the light is "reflected." The other portion will proceed in the second medium but in a changed direction, that is, the light will be "refracted" by the separating surface.

Rough surfaces reflect the light irregularly, and we speak of "reflection of diffused light." The diffused light plays an important part in nature since it enables us to see objects which receive no direct light. A perfectly flat and highly polished surface would not be visible to the eye.

2. Law of Reflection

In figure 1 is illustrated the reflection of light at (a) a plane surface, (b) a spherical convex surface, and (c) a spherical concave surface. The center of curvature of the spherical surface is designated by the letter C. A straight line from C to any point of the reflecting surface is a *"normal."* The angle i between the incident ray and the normal is called the *"angle of incidence,"* and the angle i′ formed by the reflected ray and the normal is termed the *"angle of reflection."*

The change in direction of the reflected ray at the separating surface of two media depends only on the magnitude of the angle of incidence. The reflected ray will proceed in a plane which contains the incident ray and the normal to the reflecting surface at the point of incidence. The plane determined by the incident ray and the normal is called the *"plane of incidence,"* and the plane containing the reflected ray and the normal is the *"plane of reflection."* The incident and the reflected rays always remain on the same side of the reflecting surface but on opposite sides of the normal. The law of reflection is expressed as follows: *"The angles of incidence and reflection lie in the same plane and are of equal magnitudes."*

Hence, from figure 1:

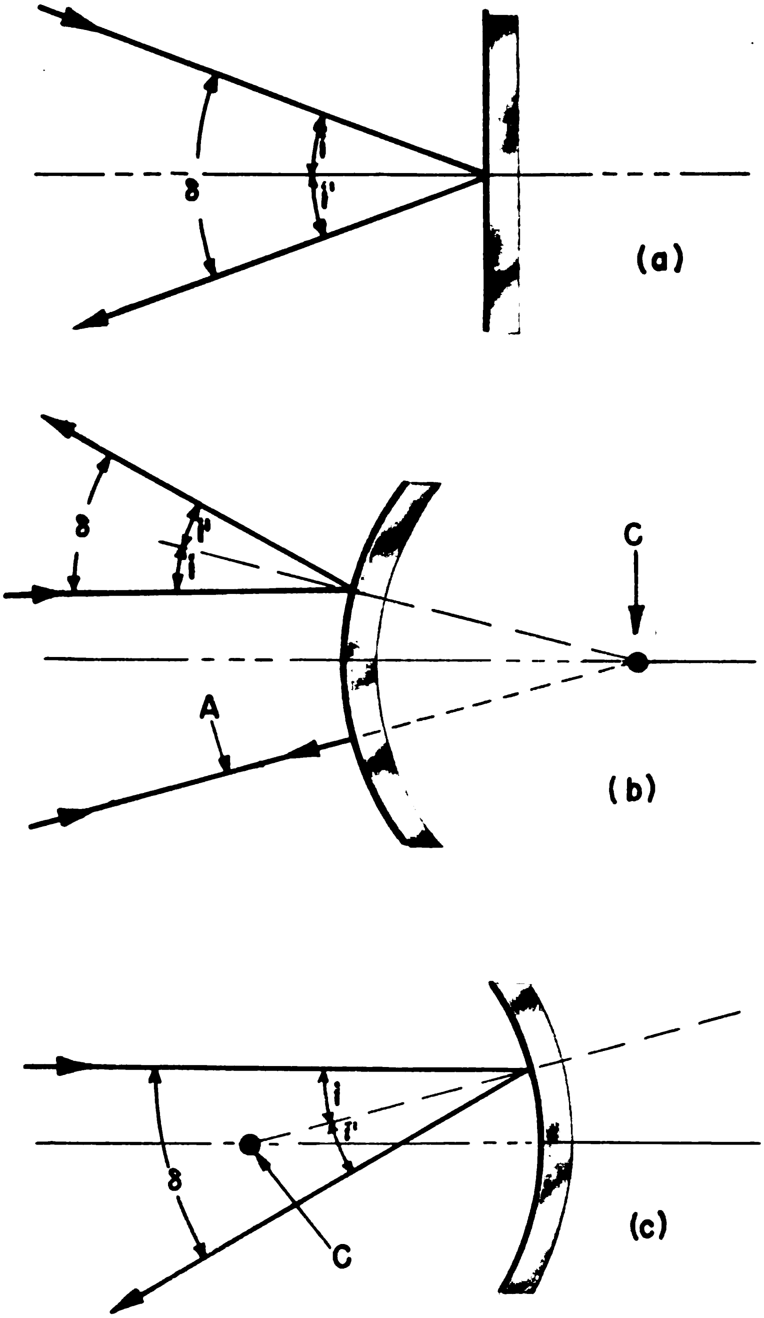
$$i' = -i \qquad (1)$$

The angles are considered positive when a clockwise turn of the ray will bring it into coincidence with the normal and negative if the turn is counterclockwise.

The angle δ included by the incident and the reflected rays is called the *"angle of deviation."* Its magnitude is twice that of the angle of incidence, hence,

$$\delta = 2i \qquad (2)$$

If the incident ray coincides with the normal as, for instance, the ray marked A in figure 1 (b), there will be no deviation and the ray is reflected back into itself.



(a) At a plane surface,
(b) At a spherical convex surface, and
(c) At a spherical concave surface.

Figure 1. Reflection of light.

3. Law of Refraction

The diagrams in figure 2 show the refraction of light rays by plane and spherical surfaces.

The angle i is the angle of incidence, and the angle i′, formed by the refracted ray and the normal is called the *"angle of refraction."* The plane determined by the incident ray and the normal is the *"plane of incidence,"* and the plane containing the refracted ray and the normal is the *"plane of refraction."* The incident and the refracted rays lie in the same plane.

The change in the direction of the refracted ray depends not only on the magnitude of the angle of incidence, but also on the refractive indices of the two media. According to the wave theory, the relative index of refraction n_{12} of two media (1) and (2) is equal to the ratio of speeds of propagation in the two media. The value of n_{12} is computed as follows:

$$n_{12} = \frac{V_1}{V_2} \tag{3}$$

where V_1 is the speed of propagation of light in the first medium and V_2 that in the second medium.

The indices of refraction of all optical media are expressed with respect to that of a standard optical medium. The medium that is selected for this purpose is that of empty space, or vacuum, and the index of refraction of a medium with respect to empty space is called the *"absolute index of refraction"* of the medium or, simply, *"the index of refraction."* The refractive index of a medium is designated by the letter n; in the case of empty space, $n = 1$. The refractive index of air, at 15° C. and under a pressure of 76 cm of mercury has been found to be equal to 1.00029; it is usually taken as equal to unity.

The law of refraction was discovered by the Dutch scientist Willebrord Snell. Snell defined the law of refraction as follows:

> *"The ratio of the sine of the angle of refraction to the sine of the angle of incidence is equal to the ratio of the refractive index of the first medium to the refractive index of the second medium."*

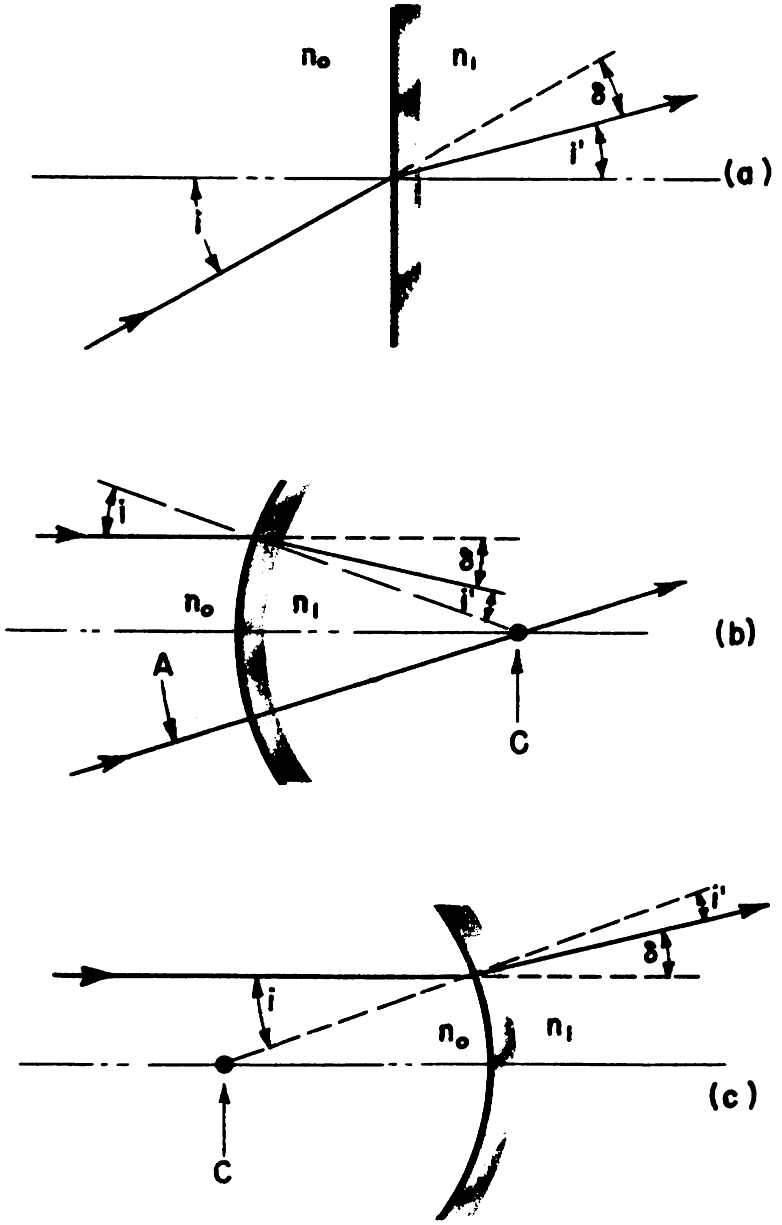
Therefore, if the refractive indices of the two media are designated by n_0 and n_1, respectively, the law of refraction is mathematically expressed thus:

$$\frac{\sin i'}{\sin i} = \frac{n_0}{n_1} \tag{4}$$

The angle of deviation is equal to the angle δ included by the incident ray produced and the refracted ray; it is equal to the difference of the angles of incidence and refraction, hence,

$$\delta = i - i' \tag{5}$$

If the incident ray coincides with the normal, there will be no refraction and the ray passes through the second medium undeviated as illustrated in figure 2 (b).



(a) By a plane surface,
(b) By a spherical convex surface, and
(c) By a spherical concave surface.

Figure 2. Refraction of light.

If the angle of incidence is given, and the refractive indices of the two media are known, then:

$$\sin i' = \frac{n_0 \sin i}{n_1} \tag{6}$$

The term n_0/n_1 is called the relative index at refraction from the first into the second medium.

The effects on the magnitudes of i′ and δ due to variations in the refractive index are shown in table I, where $n_0 = 1$ and $i = 30°$.

TABLE I

$n_1 =$	1.5	1.6	1.7	1.8	1.9
$i' =$	19°28′16″	18°12′36″	17°6′17″	16°7′39″	15°15′27″
$\delta =$	10°31′44″	11°47′24″	12°53′43″	13°52′21″	14°44′33″

4. Total Reflection

When the second medium is optically denser than the first, that is, $n_0 < n_1$, the right-hand side of equation (6) will be less than unity, and there will be always an angle i′ whose sine is equal to that value. If, on the other hand, the first medium is optically denser than the second, that is, $n_0 > n_1$, then sin i′ may be less than, equal to, or greater than unity, depending on the magnitudes of n_0 and i. For a certain value of i we find that $\sin i' = 1$, that is, $i' = 90°$. In this particular case the angle i will be designated by the letter J. This angle is called "*the angle of total reflection,*" or "*the critical angle.*" The magnitude of the angle J depends on the densities of the two media and its value is computed thus:

$$\sin J = \frac{n_1}{n_0} \tag{7}$$

If the second medium is air then $n_1 = 1$ and the above equation may be written:

$$\sin J = \frac{1}{n_0} \tag{7a}$$

The critical angle plays an important part in the design of prism telescopes and for this reason there are tabulated in table II the computed values of J for glasses whose refractive indices vary between the limits of 1.5 and 1.7.

In the diagrams of figure 3, a plane surface separates two media. Let the refractive indices of the media be $n_0 = 1.5$ and $n_1 = 1$ (air). In the first case $i = -20°$, hence i′ is smaller than the critical angle J and the ray will be refracted by the plane surface and emerges into air as shown in figure 3 (a). In the second case (see fig. 3 (b)) the angle of incidence $i = J$, hence, $i' = 90°$ and the ray emerges into air, traveling along the surface and we speak of *"grazing emergence."* In figure 3 (c) is illustrated the case when $i > J$, and we have total reflection so that $i' = -i$.

TABLE II

n	J	n	J	n	J
1.50	41°48′37″	1.57	39°33′51″	1.64	37°34′19″
1.51	41°28′19″	1.58	39°15′55″	1.65	37°18′19″
1.52	41°8′22″	1.59	38°58′17″	1.66	37°2′33″
1.53	40°48′48″	1.60	38°40′56″	1.67	36°47′3″
1.54	40°29′34″	1.61	38°23′52″	1.68	36°31′47″
1.55	40°10′40″	1.62	38°7′5″	1.69	36°16′44″
1.56	39°52′6″	1.63	37°50′34″	1.70	36°1′55″

5. Velocity of Light

The velocity of light in space is approximately 299,776 kilometers per second, or 186,272 miles per second, a velocity which would enable the light to travel in one second seven times around the earth. This velocity was first measured by the Danish astronomer Olaf Römer from the study of the appearance of the eclipse of Jupiter's satellites.

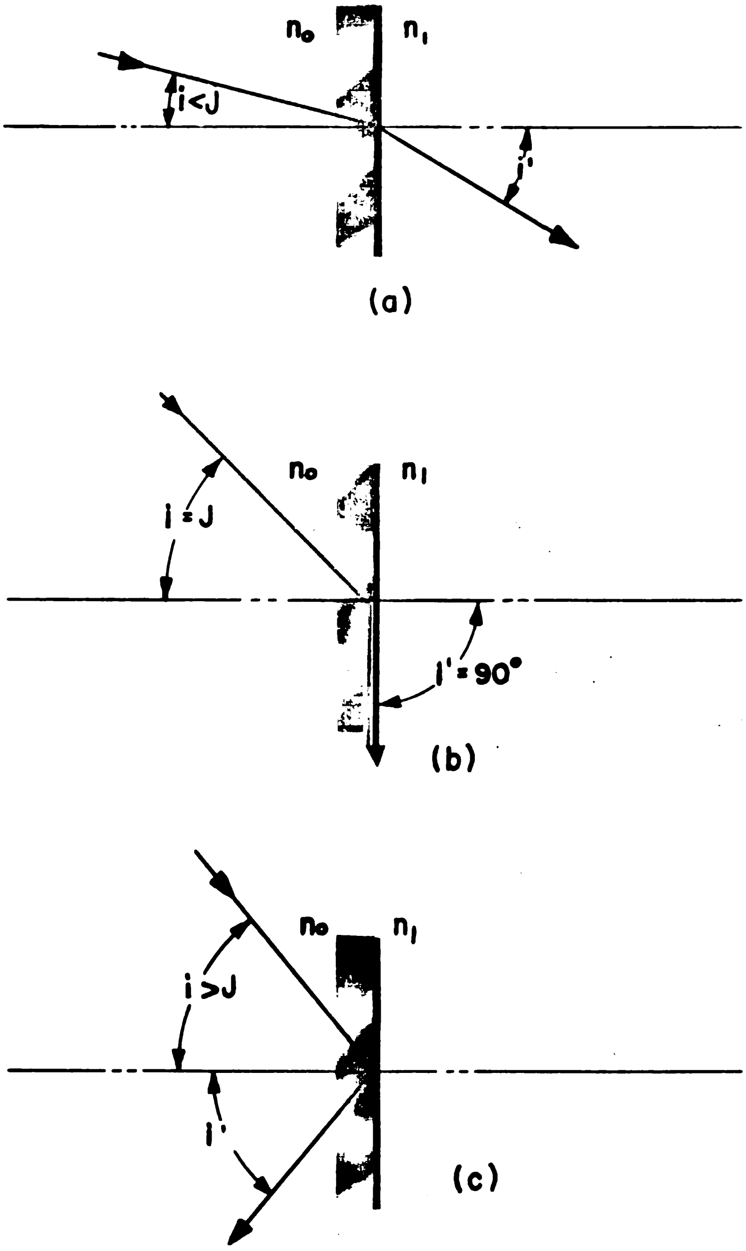
If we know the refractive index n of a medium we can easily determine the velocity V_2 of the light in that medium by means of the following formula:

$$V_2 = \frac{V_1}{n} \tag{8}$$

where V_1 is the velocity of light in space. With the aid of this equation there were computed the velocities of light in various media of known densities. The results are tabulated in table III.

TABLE III

Medium	n	Speed per second	
		Km	Miles
Space	1.0000	299,776	186,272
Water	1.3330	224,888	139,739
Fluor Crown Glass	1.4707	203,832	126,655
Turpentine	1.4744	203,321	126,337
Quartz	1.5442	194,130	120,627
Rock Salt	1.5443	194,118	120,619
Dense Flint Glass	1.9229	155,898	96,870
Diamond	2.4173	124,013	77,058



(n_0 is greater than n_1)

Figure 3. Refraction and reflection of light.

6. Dispersion of Light

The change in the direction of light refracted on a separating surface of two media depends also on the wave length λ of the light. Dispersion is due to the fact that the individual components of nonhomogeneous light, such as white light for instance, are of different colors, each of them having a different but definite wave length. The spectrum, first observed by Isaac Newton about 1670, is an image formed when a beam of light is subjected to dispersion, so that its rays are arranged in a series in the order of their wave lengths. Thus, by causing white light to pass through a prism (see fig. 4 (a)), a spectrum is obtained in which the various colors form a series from deepest red (about 770 $m\mu$ wave length) through orange, yellow, green, and blue to deepest violet (about 390 $m\mu$ wave length). The visible, or ocular, or chromatic spectrum consists of all rays which affect the eye; they lie between the limits just given. The actinic, or the chemical spectrum comprises those rays which are capable of chemical actions as, for example, in photography. These rays lie chiefly in the violet and ultraviolet regions. The thermal, or the heat spectrum, is principally composed of dark, infrared rays. Thus, the radiation we receive from the sun may roughly be divided into three parts, namely, the chemical, the luminous, and the calorific. There is no sharp dividing line between these parts, they overlap each other, the waves of each being of precisely the same nature, differing only in length.

In the spectrum of the sunlight there are seen many dark lines, the *"Fraunhofer lines,"* so called because they were first studied by Joseph Fraunhofer about 1814. Most of these lines are due to the vapors of familiar elements as, for example, the C and F lines to hydrogen, the D lines to sodium, the H and K lines to calcium. The relative positions of some of the more prominent Fraunhofer lines in the spectrum are shown in figure 4 (b).

Below are tabulated the wave lengths for light corresponding to some of the prominent lines of the spectrum.

TABLE IV[1]

1 $m\mu$ = 0.000001 mm = 0.00000003937 inches

Color of light	Line	λ in $m\mu$	λ in mm	λ in inches	Element
Red	A′	769.9	0.0007699	0.00003031	K
Red	A	759.4	0.0007594	0.00002990	O
Red	B	687.0	0.0006870	0.00002705	O
Red		670.8	0.0006708	0.00002641	L_1
Red	C	656.3	0.0006563	0.00002584	H
Yellow	D	589.3	0.0005893	0.00002320	Na
Yellow	d	587.6	0.0005876	0.00002313	He
Green		535.0	0.0005350	0.00002106	Tl
Green	E_2	527.0	0.0005270	0.00002075	Fe
Green	b_1	518.4	0.0005184	0.00002041	Mg
Green	b_2	517.3	0.0005173	0.00002037	Mg
Light Blue	F	486.1	0.0004861	0.00001914	H

[1] For references, see Bibliography, page 23.

TABLE IV—Continued

$1\ m\mu = 0.000001\ mm = 0.00000003937$ inches

Color of light	Line	λ in $m\mu$	λ in mm	λ in inches	Element
Dark Blue	G′	434.0	0.0004340	0.00001709	H
Dark Blue	G	430.8	0.0004308	0.00001696	Fe, Ca
Dark Blue	g	422.7	0.0004227	0.00001664	Cd
Violet	h	410.2	0.0004102	0.00001615	H
Violet	H	396.8	0.0003968	0.00001562	Ca

A′ = Red Potassium Line (center of double lines)
C = Red Hydrogen Line
D = Yellow Sodium Line (center of double lines)
d = Yellow Helium Line
e = Green Mercury Line
F = Blue Hydrogen Line
g = Blue Mercury Line
G′ = Blue Hydrogen Line
h = Violet Mercury Line (Bright Line)

Generally, the refractive index of a medium increases with decreasing wave length, for example, n is greater for light corresponding to the violet H line than that for the red A line. Therefore, when giving the refractive index for a medium it is also necessary to state the color of the light to which the index applies. As a rule, the letter corresponding to the Fraunhofer line in that particular section of the spectrum is used as a subscript on n as, for example, n_A, n_C, n_F, n_G, etc. In table V there are given the refractive indices for various colors of some familiar transparent media.

TABLE V[2]

Media	A	B	C	D	E	F	G′	H
Air	1.	1.	1.	1.	1.	1.	1.	1.
Water at 20°C	-------	------	1.33115	1.33300	------	1.33714	1.34035	1.3435
Rock salt	-------	------	1.54067	1.54431	------	1.5534	-------	------
Fluorite	1.43101	1.4320	1.43257	1.43393	------	1.43713	-------	------
Quartz–Ordinary	1.53917	------	1.54189	1.54424	------	1.5497	-------	------
Quartz–Extra.	1.54811	------	1.55091	1.55334	------	1.5590	-------	------
Iceland Spar–Ord.	1.6500	------	1.6544	1.6584	------	1.6678	1.6755	------
Iceland Spar–Extra	1.4826	------	1.4846	1.4864	------	1.4907	1.4943	------
Densest Flint Glass	1.89737	------	1.91038	1.9229	------	1.9545	1.9835	------
Diamond	-------	2.4074	2.4100	2.4173	------	2.4354	2.4648	------

The *"mean dispersion"* of a medium is the difference in the indices for light corresponding to the C and F lines of the spectrum. The term $n_F - n_C$ is usually designated by the symbol Δn. The Abbe constant ν is computed as follows:

$$\nu = \frac{n_D - 1}{\Delta n} \tag{9}$$

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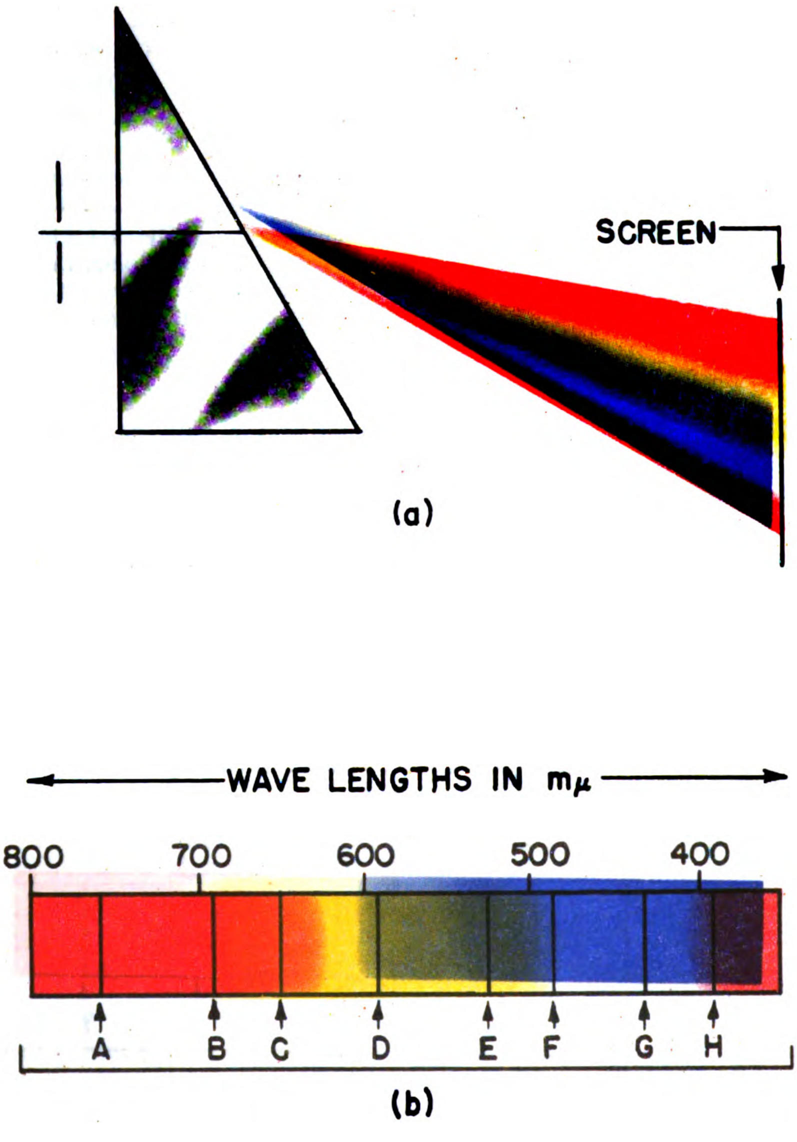
[2] For references, see Bibliography, page 23.



(a)

(b)

Figure 4. The spectrum and the Fraunhofer lines.

The numerator of this fraction is the difference between the mean index of refraction n_D of the medium and the index of refraction of air. The reciprocal of this fraction, that is, $1/\nu$, is termed the *"dispersive power,"* or *"the relative dispersion"* of the medium.

The intervals $n_C - n_{A'}$, $n_D - n_C$, $n_E - n_D$, etc., are called the *"partial dispersions"* of the glass. These intervals are given in the catalogs published by the manufacturers of optical glass. The optical designer will study the partial dispersions when selecting glasses for complicated achromatic optical systems. The partial dispersions for a few selected types of glasses, as well as their specific gravity values, are tabulated in table VI.

TABLE VI

Type of glass	n_D	ν	$n_F - n_C$	$n_C - n_{A'}$	$n_D - n_C$	$n_g - n_F$	Sp. Gr.
Fluor Crown	1.47076	67.2	.00700	.00253	.00216	.00373	2.30
Boro Silicate	1.51641	64.0	.00806	.00287	.00248	.00432	2.50
Barium Crown	1.57259	57.5	.00996	.00341	.00303	.00545	3.23
Dense Crown	1.61034	56.5	.01080	.00369	.00327	.00592	3.58
Barium Light Flint	1.54748	53.6	.01021	.00346	.00307	.00566	2.98
Ordinary Flint	1.62016	36.3	.01706	.00545	.00500	.00995	3.66
Dense Flint	1.64786	33.9	.01912	.00604	.00559	.01127	3.96
Special Dense Flint	1.92322	20.9	.04408	.01301	.01248	.02775	6.67

The diagrams in figure 5 illustrate the dispersion of light on a plane surface, a spherical convex and a spherical concave surface. A ray of white light is incident at the refracting surface, making the angle i with the normal. At the point of incidence it will be resolved into a pencil of colored rays because the index of refraction depends on the color of the light. The red rays will be refracted the least, the blue rays the most. In order to determine the magnitude of the angle of refraction for the various colored rays we write equation (6) as follows:

$$\sin i'_\lambda = \frac{n_0 \sin i_\lambda}{n_\lambda}$$

where n_λ assumes successively the values of $n_{A'}$, n_C, n_D, etc., and we will obtain a different value in each case. The angle included by the refracted blue ray (g) and the refracted red ray (A') is equal to $\delta_g - \delta_{A'}$ and it is called the *"angular chromatic aberration."* In the case of the spherical convex surface the refracted rays will intersect the axis at various points designated by the letters g, F, D, C, and A', respectively. In the case of the spherical concave surface the refracted rays, produced backwards, will intersect the axis as shown in figure 5(c). The distance between the points g and A' is termed *"longitudinal chromatic aberration."* In the case of instruments for visual use, such as telescopes for instance, we consider light corresponding to the C and F lines of the spectrum. In this case the longitudinal chromatic aberration is equal to the distance between the points C and F, while the angular chromatic aberration is equal to $\delta_F - \delta_C$. Comparing the intersection distances of the convex with those for the concave surface gives us the cue as to how to obtain an achromatic lens, namely, by combining a negative lens with a positive lens. This will be explained in a later section.

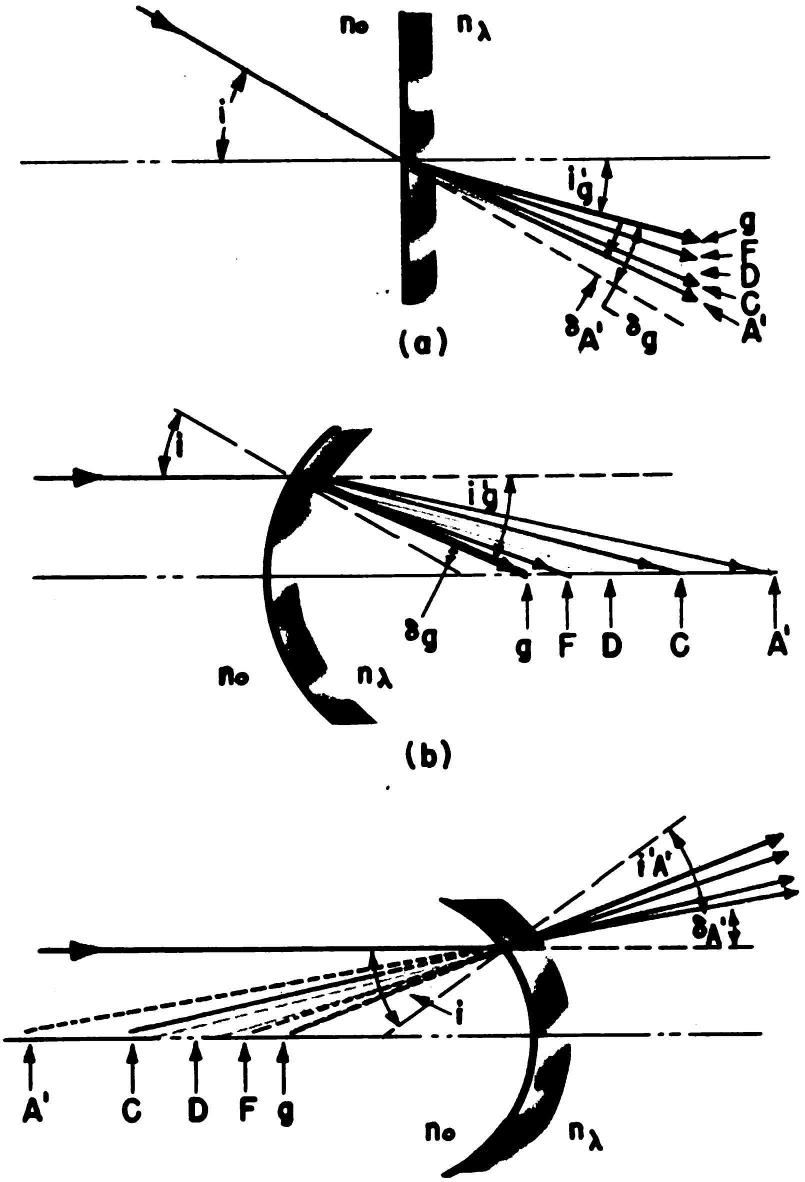


Figure 5. Dispersion of light (chromatic aberration).

7. Atmospheric Refraction

A ray of light proceeding from the sun S (see fig. 6 (a)) will travel toward the earth in a straight line while in empty space. When approaching the earth the path of the ray will be changed gradually due to the differences in density of the air, and an observer at O on the earth's surface will see the sun in the position indicated by S′. The angular height of the sun, that is, the angle included by the tangent to the surface of the earth at the point O and a straight line from O to the true position of the sun, is designated by the letter α. The angle of deviation, that is, the angle between the two straight lines O—S and O—S′, is denoted by the letter δ.

When the sun is in the zenith, a position indicated by the letter Z in figure 6(b), the angle α is now equal to 90° and $\delta = 0$. Hence, this is the only position where the sun will appear in its true position. For points near the horizon the refraction is so great that the rising sun or moon may be observed while they are still below the horizon, a case also illustrated in figure 6 (b). The mean values of δ for various angular heights α were determined by W. Weiler[3] and are given in table VII.

TABLE VII

α	δ	α	δ	α	δ	α	δ	α	δ
0°	33′40″	3°	14′28″	10°	5′20″	40°	1′9″	80°	0′10″
1°	24′21″	5°	9′54″	20°	2′39″	60°	0′34″	90°	0′0″

The question arises, what is the extreme distance D at which an object can be seen? Referring to the diagrams in figure 7 (a), let O be the position of an observer at a distance H above the earth's surface, and 2R is the diameter of the earth (2R=7,899.6 miles, or 13,903,414 yards), then we derive easily:

$$D = 2R \sin \frac{\alpha}{2}, \qquad \sin \frac{\alpha}{2} = \sqrt{\frac{H}{2(H+R)}}$$

hence,

$$D = R \sqrt{\frac{2H}{H+R}} \tag{10}$$

In most cases H is small as compared to R and we have:

$$D = \sqrt{2HR} \tag{10a}$$

If H is given in feet and D is expressed in yards, then,

$$D = 2153 \sqrt{H} \tag{10b}$$

If the target T is elevated above the surface of the earth by the distance h (see fig. 7(b)) the distance D is then determined as follows:

$$D = 2R (\sqrt{H} + \sqrt{h}) \tag{11}$$

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[3] For reference, see Bibliography, page 23.

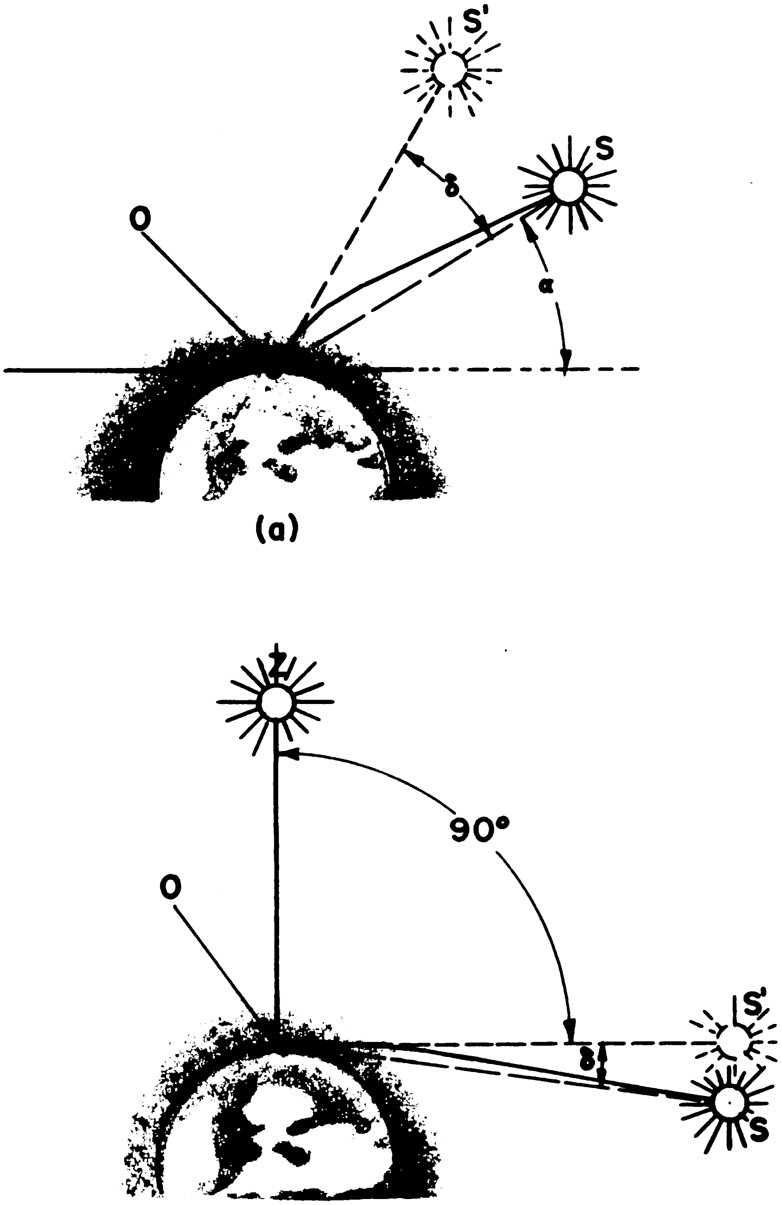

Figure 6. Atmospheric refraction.

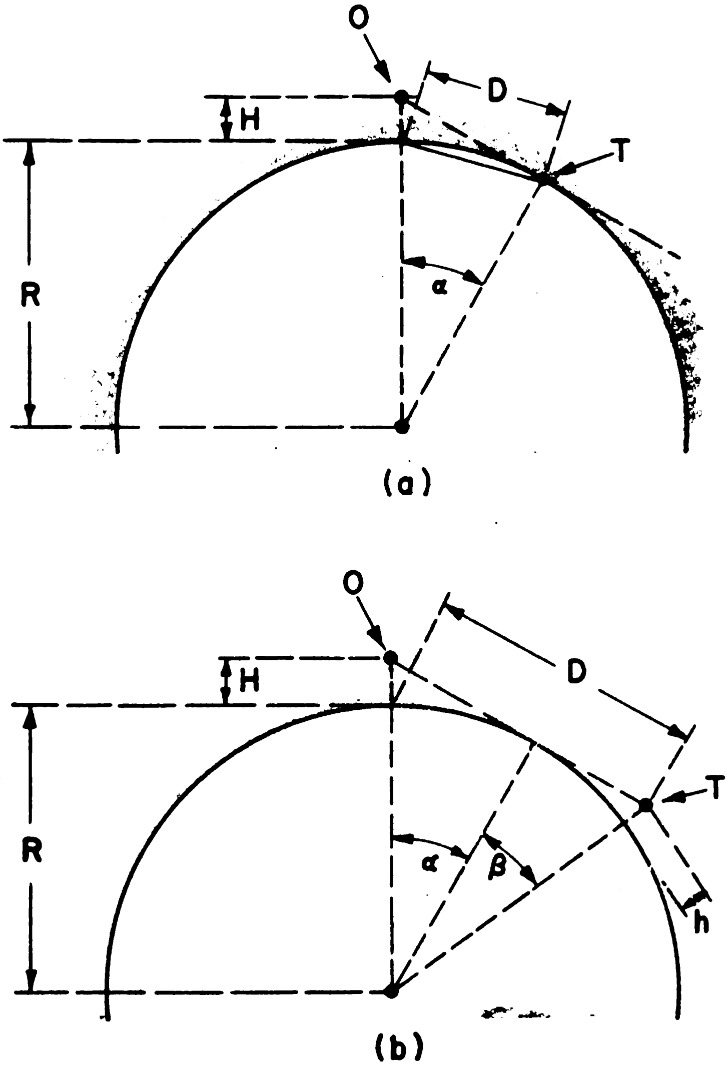

Figure 7. Range of vision.

or, when h and H are given in feet and D is expressed in yards we may write the above equation:

$$D = 2153\ (\sqrt{H} + \sqrt{h}) \tag{11a}$$

In the case of atmospheric refraction objects appear farther above the horizon, and the maximum at which objects can be seen increases. Therefore, if we consider atmospheric refraction we must multiply the value of D by a factor m. The value of this factor is commonly assumed to be equal to:

$$m = \sqrt{\frac{7}{6}} = 1.08012 \tag{12}$$

$$D' = mD \tag{13}$$

where D′ is the maximum distance that can be seen when atmospheric refraction is considered.

The values of D given in table VIII are more often correct when the observation is made over land due to the absence of water vapor in the atmosphere.

In the case of the depression position finder, which determines the range by measuring the angle from the point of observation to the target, the results would be useless in most cases if the instrument were not provided with some compensating device which will correct the errors introduced by atmospheric refraction. In table VIII are tabulated the values of the maximum distances at which objects can be seen in which D is the distance (in yards) with no refraction and D′ with refraction. H is the height (in feet) of the observer's eye above the surface of the earth and h is height (in feet) of the object above the surface.

TABLE VIII

	h = 0		h = 50	
H	D	D′	D	D′
10	6,808	7,353	22,033	23,795
25	10,765	11,627	25,988	28,070
50	15,224	16,444	30,448	32,887
75	18,646	20,140	33,870	36,584
100	21,530	23,255	36,754	39,699
200	30,448	32,887	45,672	49,331
300	37,291	40,279	52,515	56,723
400	43,060	46,510	58,284	62,954
500	48,143	52,000	63,367	68,444
600	52,738	56,963	67,962	73,407
700	56,963	61,527	72,187	77,971
800	60,896	65,775	76,120	82,219
900	64,590	69,765	79,814	86,209
1,000	68,084	73,539	83,308	89,983

The results compare favorably with those given by I. C. Gardner.[4] It is stated that more correct values for D are obtained with Kohlschütter's formula:

$$m = 0.9463 + 0.00036\ (t - 15^\circ) - [0.1922 - 0.00131\ (t - 15^\circ)]\frac{\Delta}{\sqrt{H}} \qquad (14)$$

where t is the temperature of air, and Δ is the difference in the temperature of the air at one meter above the surface of the water and the temperature of the water on the surface.

8. Phenomena Due to Reflection, Refraction, and Dispersion of Light

A natural phenomenon due to reflection, refraction, and dispersion of light in drops of water falling through the air is the rainbow.

The rainbow is seen as an arch of light on the horizon, appearing in all colors of the spectrum, usually at the close of a shower. The outer color band of the rainbow is red while the inner band is violet. Between the two there appear the bands of the remaining colors of the solar spectrum. These bands are not sharply separated but they overlap each other, thus creating additional shades of color. This type of rainbow is known as the primary rainbow and is caused by light suffering one reflection in a raindrop as shown in figure 8(a).

Sometimes there are visible two concentric rainbows, the smaller inner one being the primary rainbow just described, and the larger outer one known as the secondary rainbow. The outer rainbow is never as brilliant as the inner one, and the order of the color bands is reversed; that is, the outer band is violet while the inner band appears red as illustrated in figure 8(c). The secondary rainbow will be visible when the light enters the raindrops in the upper regions of the air, and then suffers two reflections before emerging from the drop as shown in figure 8(b).

Another natural phenomenon due to reflection and refraction of light is known as the "Fata Morgana." It is frequently observed from the hills near the city of Reggio in the southern part of Italy.

Mirages of this kind are also common occurrences in the desert countries and are explained thus: The upper strata of the air will be cooler than the stratum touching the hot sands and, therefore, the density of the air will gradually decrease when approaching the sands. Consequently, a ray of light proceeding from a distant object in a downward direction will be refracted away from the normal until it reaches a point in the lower strata where it will be totally reflected back, and proceed in an upward direction. At each layer of air it will be refracted towards the normal until it will finally enter the eye of the observer as illustrated in figure 9. The observer will see in the sands an inverted image of the palm.

Mirages are also frequently observed in the arctic regions. There the air is cooler and denser in the lower strata than the air in the upper strata. A ray of light proceeding from an object, say a boat, will travel in an upward direction. Upon incidence at each of the higher strata it will be refracted away from the normal until it reaches a

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[4] For references, see Bibliography, page 23.

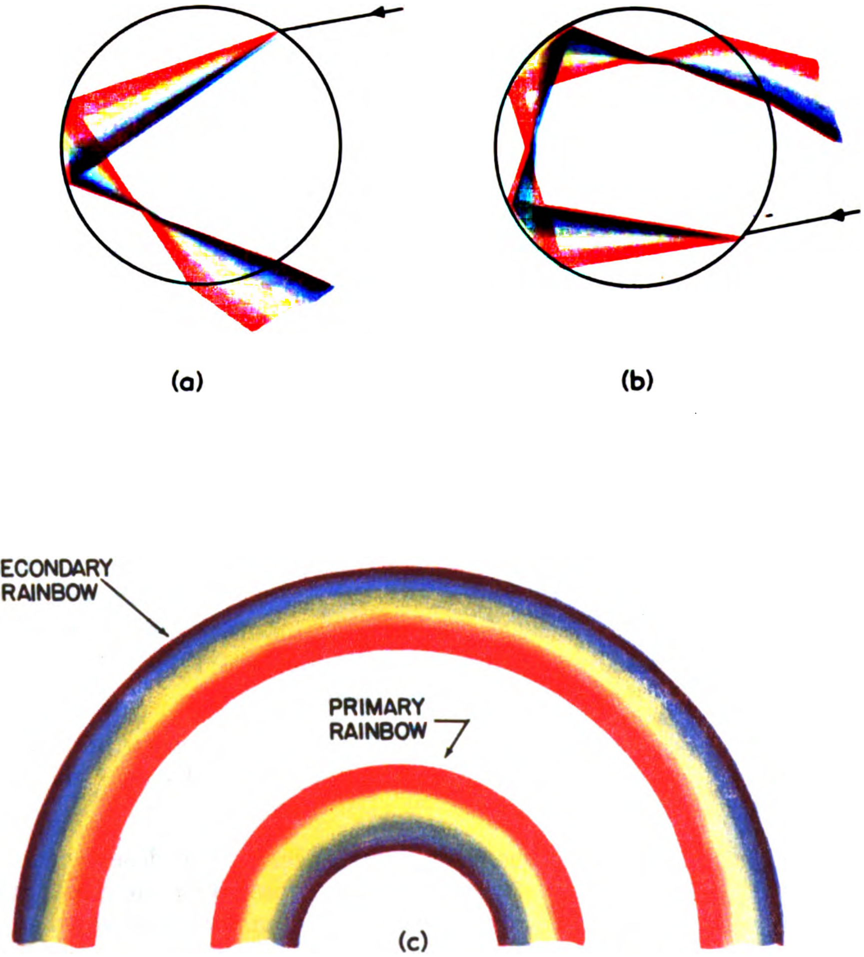

Figure 8. The rainbows.

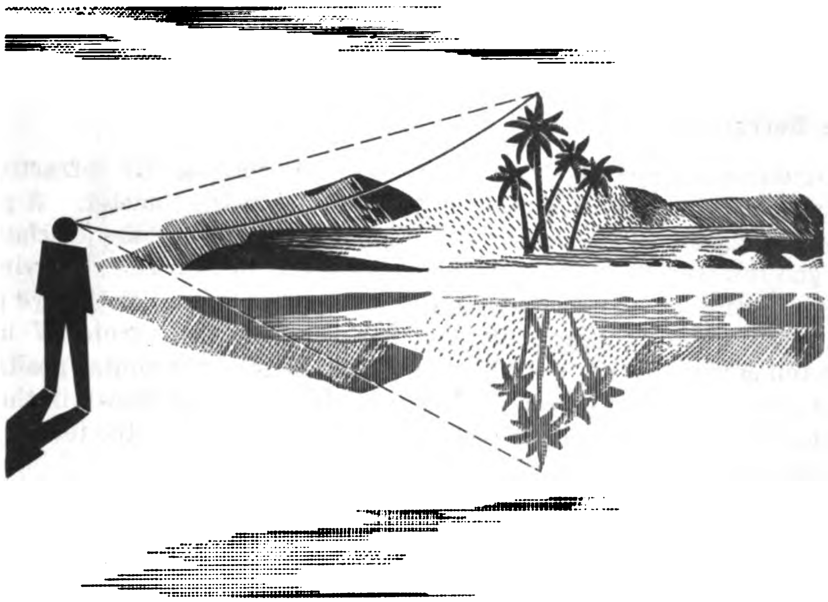

Figure 9. Mirages seen in the desert countries.

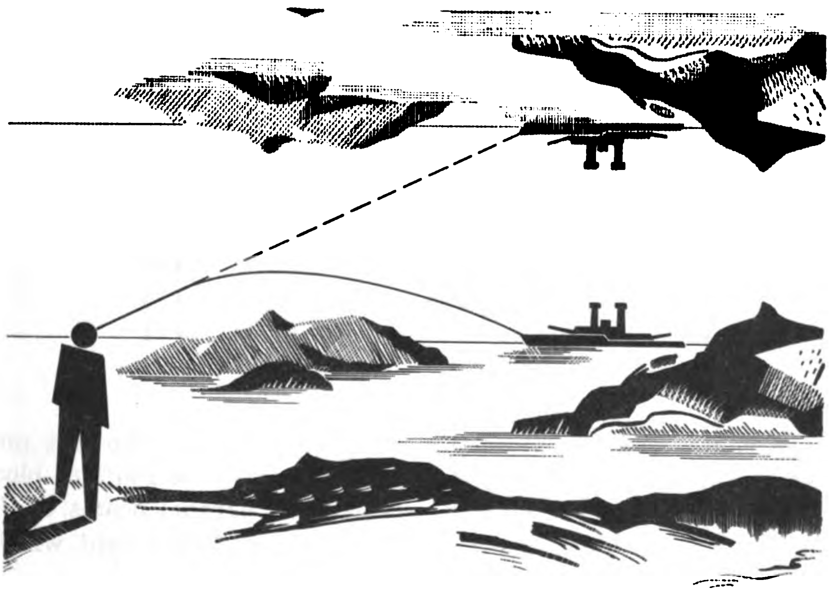

Figure 10. Mirages seen in the polar regions.

point in the upper strata where it will be totally reflected. Proceeding, it will travel in a downward direction until it will enter the eye of the observer as shown in figure 10. The observer will see in the air an inverted image of the boat.

9. Measuring Refractive Index n

The most convenient and accurate method of determining the refractive index of a medium experimentally is with the aid of the Pulfrich refractometer. A plan view of this instrument is shown in figure 11(a). Light coming from the discharge tube T will be converged towards the test piece by the lens system L. The observing telescope OT houses the objective with cross hairs in its focal plane, as well as the eye piece. One fourth of the circle C is graduated, and for fine readings the vernier V is used. In order that the top surface of the Pulfrich block remain in a horizontal position there is placed in front of the telescope objective a right-angle prism (not shown in the diagram), which will reflect the light in a horizontal direction. In that case the telescope is placed in a vertical position.

The Pulfrich block is made of a very high refracting glass whose refractive index will be designated by the letter N. The top surface of the block and the surface nearest the observing telescope are optical flats.

The solid test piece whose refractive index n is to be determined must have two polished surfaces which are at nearly right angles. The surface which is brought into contact with the Pulfrich block must be a fairly good optical flat.

A layer of liquid, whose refractive index is higher than that of the test piece, is placed between the Pulfrich block and the test piece. It is advantageous to make this difference in the refractive indices small by selecting a suitable liquid which has a refractive index as near as possible to that of the test piece.

In table IX are tabulated six liquids suitable for use with the Pulfrich refractometer.

TABLE IX

Liquid	n_D
Aniline	1.59
Quinoline	1.62
Monobromonaphtaline	1.66
Mercuric Potassium Iodide	1.72
Methylene Iodide	1.74
Mercuric Barium Iodide	1.79

The light, emerging from the discharge tube T, will enter the test piece slightly converging as shown in figure 11 (b). After emerging from the Pulfrich block the light will enter the observing telescope, which is focused for parallel beams of light. As a rule, monochromatic light is used during measurements and this light will be confined to one side of a sharp defined line in the field of the observing telescope. The sharp dividing line marks the limiting angle. The width of the patch of light beyond the line depends on the angle of convergence of the incident beam of light.

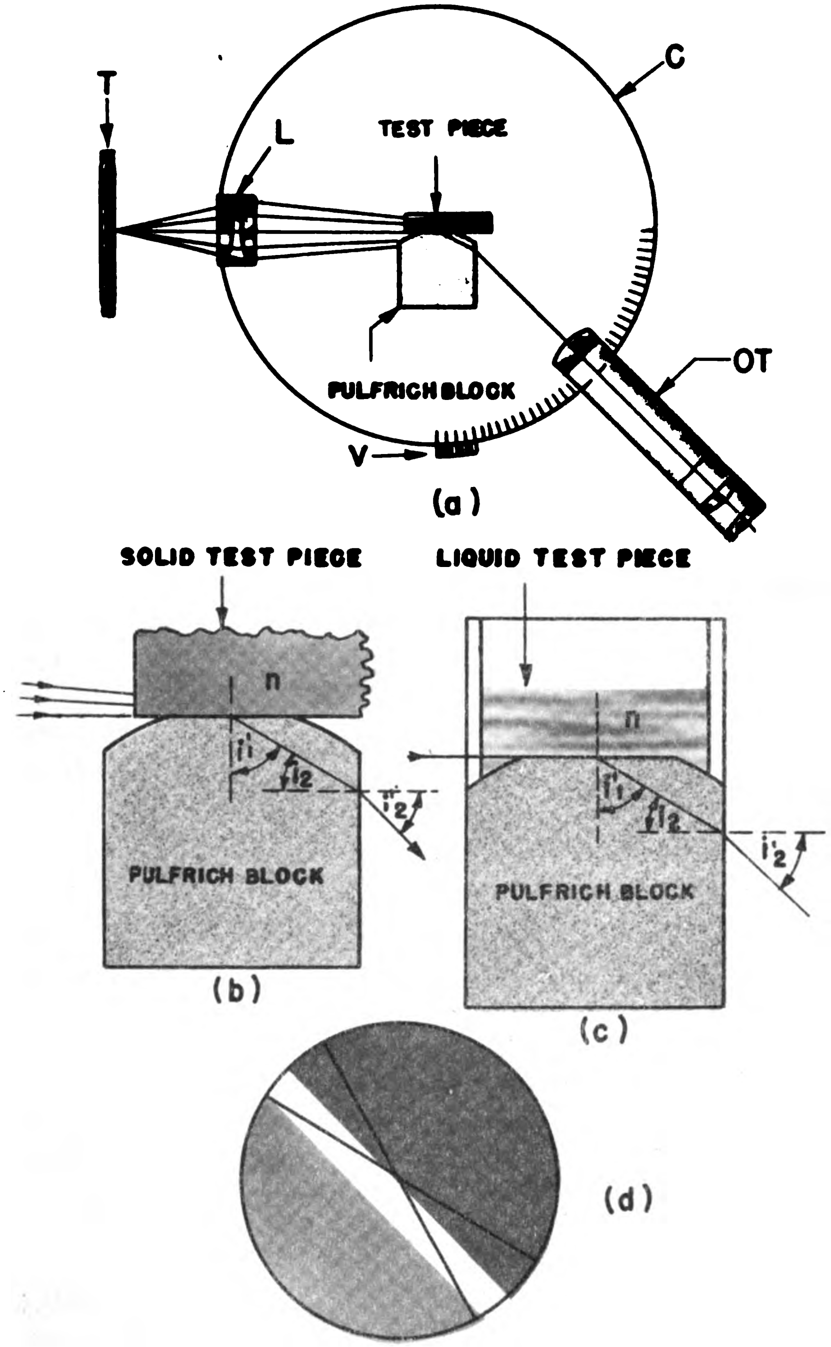

Figure 11. Pulfrich refractometer.

The lower incident ray will graze the lower surface of the test piece, and, upon entering the Pulfrich block, it will make the angle i_1' with the normal of the top surface. Proceeding, the ray will make the angle i_2 with the normal to the exit surface of the block and, upon emerging from the block it will include the angle i_2' with the normal. It is the magnitude of this angle which will determine the value of the refractive index n of the test piece. In order to determine the exact value of the angle i_2' move the observing telescope until the intersection of the cross hairs in the telescope falls on the sharp dividing line in the field as illustrated in figure 11(d). Read the value of i_2' on the vernier V.

Knowing the values of N and i_2' we are able to compute the value of the refractive index with the following equation:

$$n = \sqrt{N^2 - \sin^2 i_2'} \tag{15}$$

As a rule, there are furnished tables with each Pulfrich refractometer which give the values of n for different values of N and i_2'.

If the refractive index of a liquid is to be determined, a small glass cylinder is cemented to the curved portion of the top surface of the Pulfrich block, and the liquid to be tested is poured into the cylinder as illustrated in figure 11 (c).

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SECTION II

REFLECTIONS AND REFRACTIONS BY MIRRORS

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10. Thin Mirrors

For the sake of simplicity it is assumed that the front surface is silvered so that all reflections take place on this surface. In the diagrams the mirrors are shown perpendicular to the plane of the paper, the lines showing their traces in this plane. Therefore, the plane of incidence and the plane of reflection coincide with the plane of the paper.

11. Plane Mirror

A ray of light proceeding from the luminous point P will be reflected by the mirror M at the point C in the direction of c (fig. 12). If this ray is produced backwards, it will intersect a straight line drawn from P perpendicular to the mirror at the point P′ as illustrated. The points P and P′ are a pair of conjugate points with respect to the mirror. The point P′ is the "*Virtual Image*" of the luminous point P. It is termed virtual image because the rays of light do not actually pass through the point P′ but only appear to do so.

The position of the image-point P′ is independent of the position of the point of incidence C, hence, all rays proceeding from the object-point P will be reflected along paths which, when produced backwards, will intersect at the point P′. Therefore, if we place the eye in any of the positions indicated by the letters a, b, c, etc. in figure 12, the image appears always at the point P′. The plane mirror is the only optical system which fulfills the geometrical condition that to every object-point there shall correspond one, and only one, image-point.

The straight line P—P′ is bisected at right angles by the reflecting surface of the mirror, hence, object and image distances are equal, or,

$$s' = -s \tag{16}$$

If the object is an extended one as, for example, the object P—p in figure 13, we find the position and the size of the image by drawing from P and p straight lines normal to the reflecting surface of the mirror M. The conjugate points P′ and p′ are located on these lines at a distance of $s' = -s$ behind the reflecting surface.

P′—p′ is the image of the extended object P—p, and it is equal in size to that of the object. However, the image will appear "*reversed*" or "*reverted,*" that is, to the eye of the observer at E the tip of the pennant will appear to the left. If we assume the mirror removed and the eye placed at E′ the tip of the pennant will be seen to the right, that is, in its normal position.

The "*REVERSION*" or "*REVERTION*" of the image is clearly illustrated in figure 14. Printed characters are upright but appear to run from right to left.

The extent of the area of the mirror that is actually utilized by the rays entering the eye of the observer will depend on the size and the position of the object viewed, and

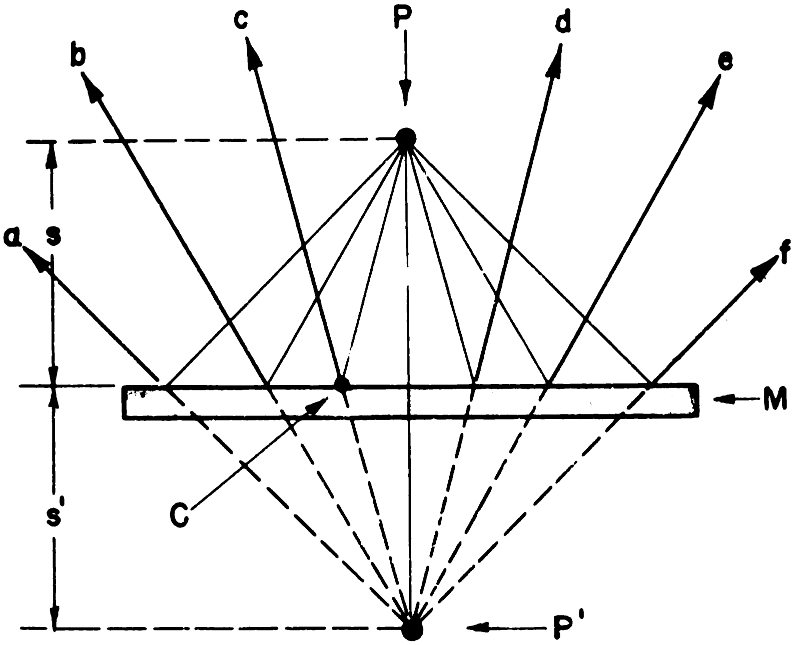


Figure 12. Image of a point object.

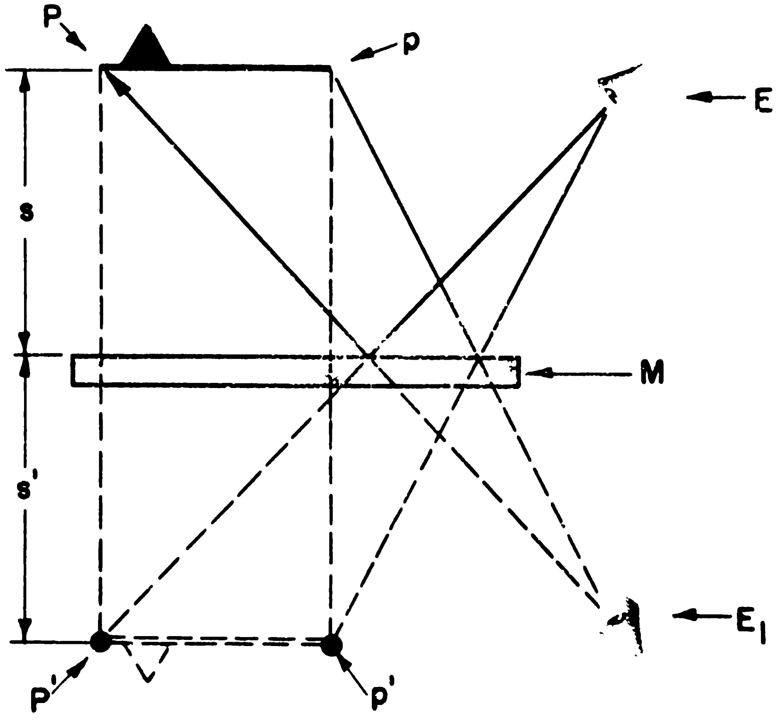
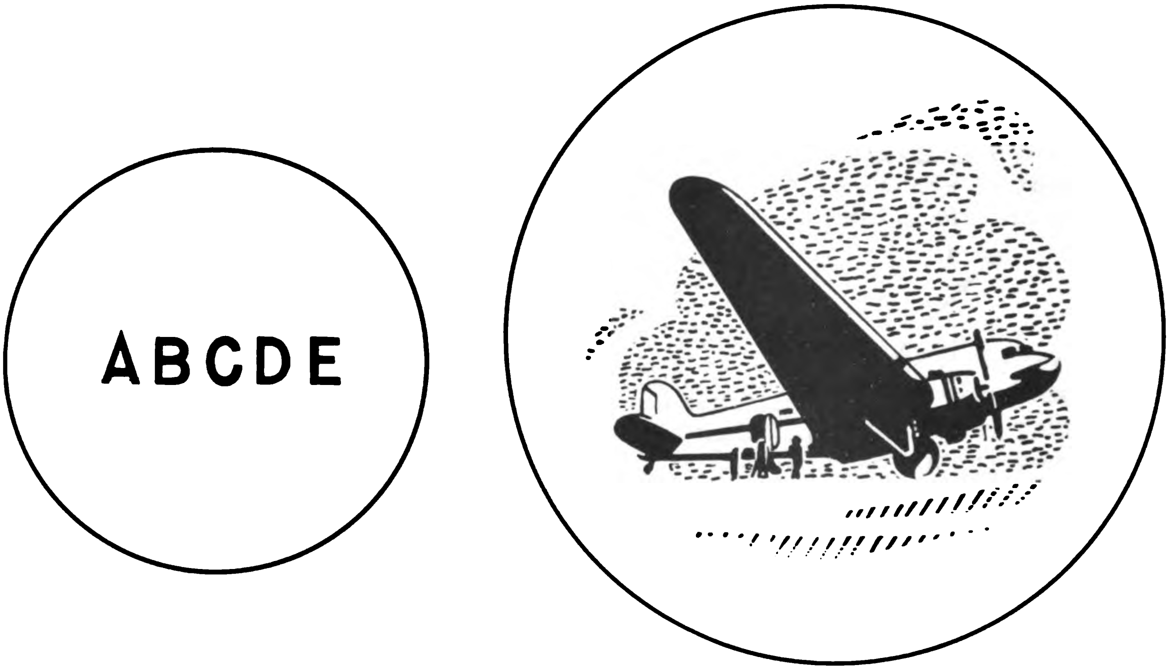
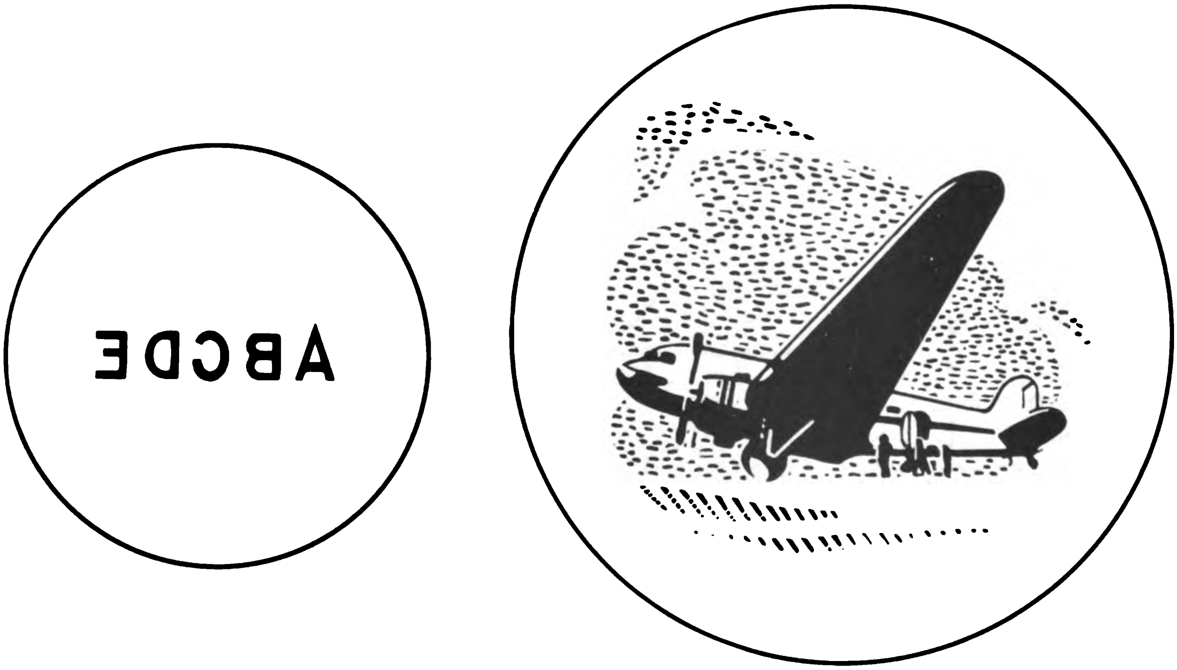


Figure 13. Image of an extended object.



NORMAL IMAGE



REVERTED IMAGE

Figure 14. Normal and reverted images.

also on the position of the observer's eye. If the object and the mirror are parallel to each other, and if the eye of the observer is situated at a distance d from the mirror we can determine the length L of the portion of the mirror (see fig. 15) which is actually utilized by the rays with the following equation:

$$L = \frac{Ad}{d + s} \tag{17}$$

where A is the height of the object.

If the observer desires to view his image from head to foot, then $d = +s$ and equation (17) may be written thus:

$$L = \frac{A}{2} \tag{17a}$$

Therefore, in order that an observer standing erect in front of a vertical mirror may be able to view his image from head to foot, the length L of the mirror must be at least half the height A of the observer, and the lower edge of the mirror must be placed at a level half way between the levels of the eye and feet. This condition is illustrated in figure 16.

12. Rotating Plane Mirror

A number of the most important uses of the plane mirror depend on the fact that when a mirror is turned through a given angle about an axis perpendicular to the plane of incidence, the reflected ray will be turned through an angle just twice as great.

The ray P—C (see fig. 17) proceeding from the luminous point P will be incident normally at the reflecting surface of the mirror M—m and, therefore, it will be reflected back into itself. We now turn the mirror counterclockwise about the axis C through an angle α. The mirror will now be in the position M_1—m_1. When turning the mirror through the angle α we also turn the normal to the reflecting surface through the same angle and, consequently, the angle of incidence for the ray P—C will also be equal to α. According to the law of reflection the angle of reflection is equal to the angle of incidence; hence, the angle of deviation, that is, the angle between the incident ray P—C and the reflected ray C—P′ will be equal to 2α.

The designs of a great number of scientific instruments are based on this principle as, for example, the Heliograph, the Galvanometer by Poggendorf, the Mirror Sextant, and many others.

The Mirror Sextant is an instrument used for measuring angles between distant objects. The essential parts of the mirror sextant, designed by Hadley in the eighteenth century, are shown in figure 18. The mirror M_1 can rotate about an axis perpendicular to the plane of incidence. The mirror M_2 is in a fixed position so that its reflecting surface and the line of sight will include an angle of 60°. The lower half of this mirror is silvered to prevent the light from passing through this portion of the mirror. An index arm is attached to the mirror M_1 which points to the number on a scale. This number indicates the magnitude of the angle δ through which the mirror has deviated the light.

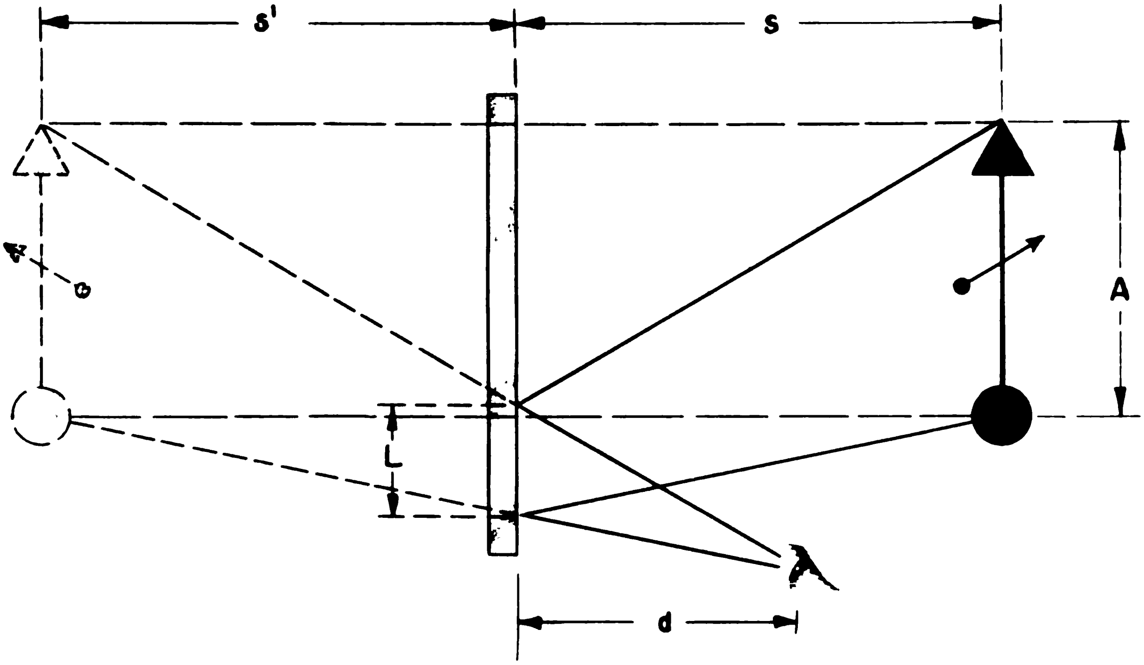


Figure 15. Area of mirror utilized by the rays.

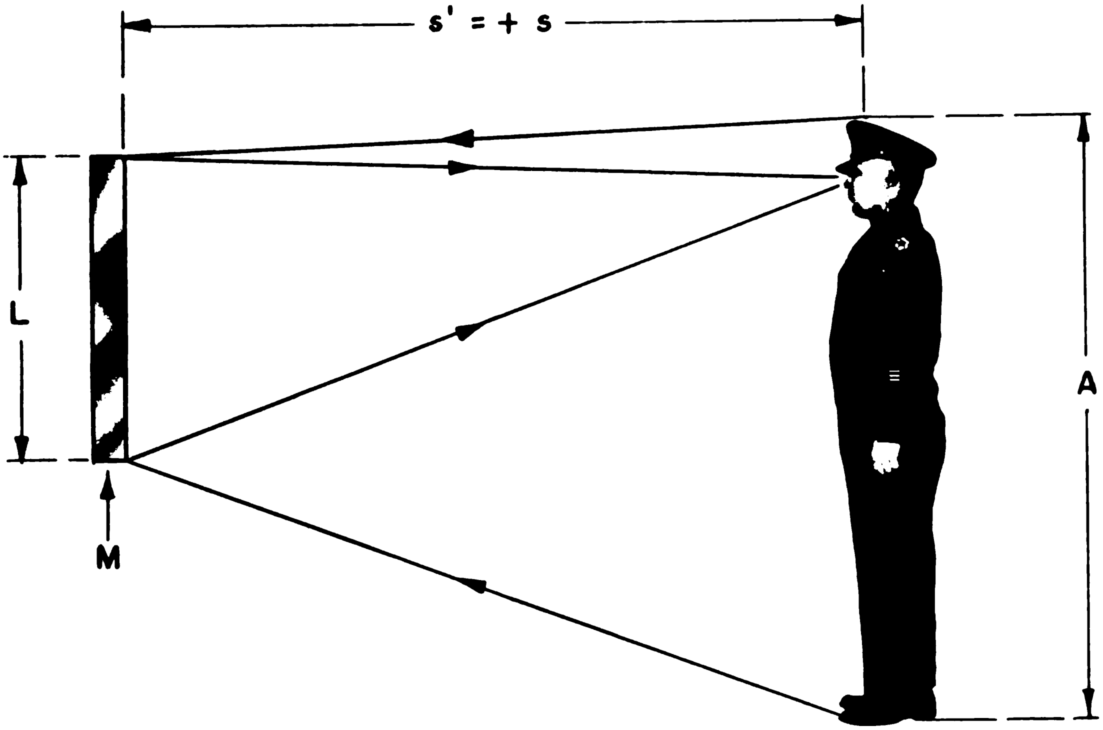


Figure 16. Size of mirror so that the observer can view his image from head to foot.

Neither the index arm nor the scale are shown in the diagrams. The rays A and B come from a distant target, say a lighthouse, on the horizon. If the mirrors are parallel to each other as shown in figure 18a, the two rays will be in coincidence and the eye will see one image of the lighthouse. In this case the index arm will point to the figure 0 since there is no deviation. We have:

$$i' = -i \text{ or } i_1 = -i_1' = 30°$$

If it is required to determine the angle between the lighthouse and the sun, then the mirror M_1 must be rotated clockwise through an angle α until the images of the lighthouse and the sun are in coincidence. In figure 18b is shown the ray B coming from the sun, and the mirror M_1 turned through an angle of $\alpha = 30°$; hence, the angle through which this ray is deviated is equal to $\delta = i = -i = 2\alpha = 60°$. The index arm will now point to 60 on the scale. The modern sextants are provided with a telescope which is placed between the mirror M_2 and the eye.

13. Two Parallel Plane Mirrors

If the luminous object O (see fig. 19) is placed between the two parallel plane mirrors M_1 and M_2 there will be formed two series of images by successive reflections. The images, infinite in number, are all situated on the normal of the two reflecting surfaces passing through the object O. Object and images are equal in size and are separated by a distance equal to the distance between the two reflecting surfaces. The images formed after an even number of reflections are similar to the object and the others are reversed. Since at every reflection some light is lost, the images situated at great distances from the object are hardly visible.

14. Two Intersecting Plane Mirrors

The luminous object O placed between the two intersecting plane mirrors M_1 and M_2 will give rise to two series of images. Images similar to the object will alternate with reversed images. The number N of the images depends on the magnitude of the angle α included by the two reflecting surfaces. The distance of each image from the center point C (see fig. 20) is equal to that of the object from C and, therefore, the images lie on the circle A. The positions of the various images are determined as follows:

Draw a straight line from the center of O perpendicular to the mirror M_1 through to the circle A. At the point of intersection of this line and the circle is found the first image O_1. Draw a straight line from the image-point O_1 perpendicular to the reflecting surface of M_2 and find the image-point O_2 at the point of intersection of this line and the circle A. Again, draw a straight line from the image-point O_2 normal to the mirror M_1 and find on the circle the image-point O_3, etc. Likewise, draw a straight line from O perpendicular to the mirror M_2 and find the image-point O_1' on the circle A. Continue in this manner until all the image-points have been located.

If the angular distance from the center of the object O to the reflecting surfaces of M_1 and M_2 are designated by the letter β, then, $\alpha = 2\beta$. The total number N of images that can be seen is determined as follows:

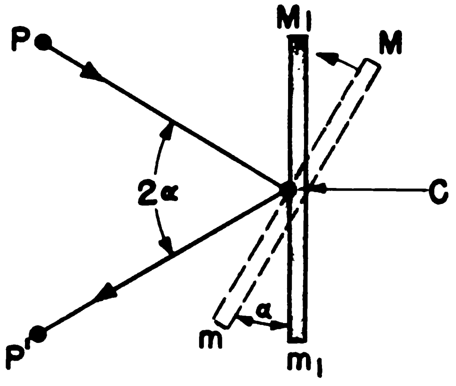


Figure 17. The rotating plane mirror.

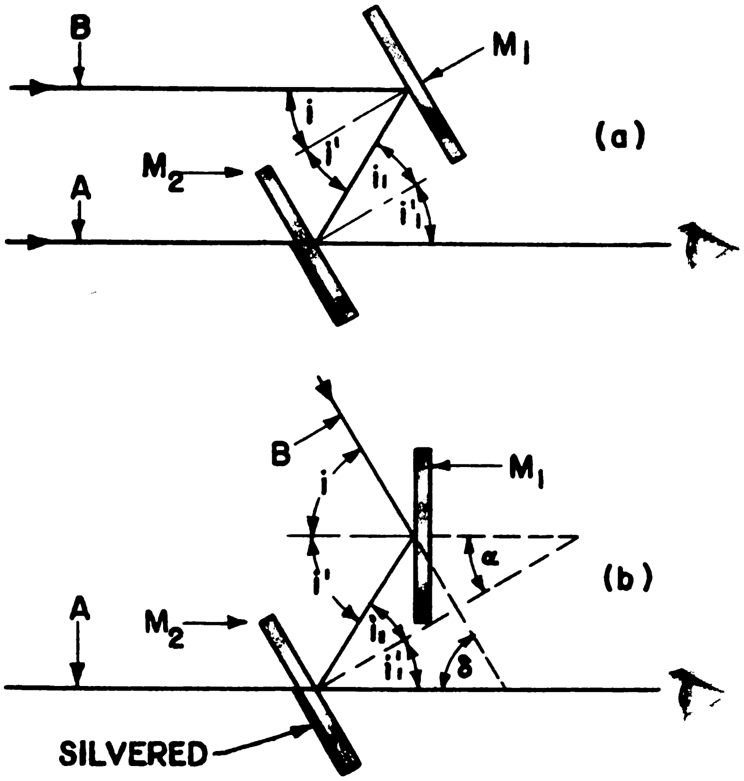


Figure 18. The optical system of a mirror sextant.

a. If the angle α is contained a whole number of times, say m times, in 180°, that is, $m = 180°/\alpha$, then the number of images that can be seen will be equal to:

$$N = 2m - 1 \tag{18}$$

For example, in figure 20 the angle α is equal to 45° and hence $m = 4$ and $N = 8 - 1 = 7$.

b. If the angle α is contained in 180° m times with a remainder p so that $180°/\alpha = m + p/\alpha$, the number N of images seen are determined as follows:

(1) $N = 2m + 2$ when $p > \alpha/2$

(2) $N = 2m + 1$ when $p = \alpha/2$

(3) $N = 2m$ when $p < \alpha$ and $< \beta$

c. If the angle α becomes zero, then $N = \infty$ and this expresses the condition of the two parallel plane mirrors discussed in the previous paragraph.

On the principle of the two intersecting plane mirrors are based the designs of various scientific instruments as, for example, the Kaleidoscope by Brewster, the Heliotrope by Gauss, and others. In the case of the Kaleidoscope, two long narrow plane mirrors including an angle of 60°, are mounted in a tube. The lower end of the tube is sealed with a polished glass disc and a frosted glass disc. Between these discs are placed small chips of colored glasses, or colored beads, etc. When looking into the tube from the upper end there will be seen beautifully colored designs. The shapes of these designs can be changed by rotating the tube during observations.

15. Concave mirror

In this and the subsequent paragraphs there will be used the following abbreviations:

A = the height of the object P—p.

A′ = the height of the image P′—p′. A minus (—) sign before A′ indicates that the image is inverted.

C = the center of curvature of the reflecting surface.

f = the focal length of the mirror M.

F = the focal point of the mirror M.

R = the radius of curvature of the reflecting surface. It is considered positive (+) when convex toward the incident light, and vice versa.

s = the object distance, that is, the distance from the reflecting surface of the mirror to the object. It is reckoned negative when the object is located to the left of the mirror, and vice versa.

s′ = the image distance, that is, the distance from the reflecting surface of the mirror to the image. It is considered positive when the image is located to the right of the mirror, and vice versa.

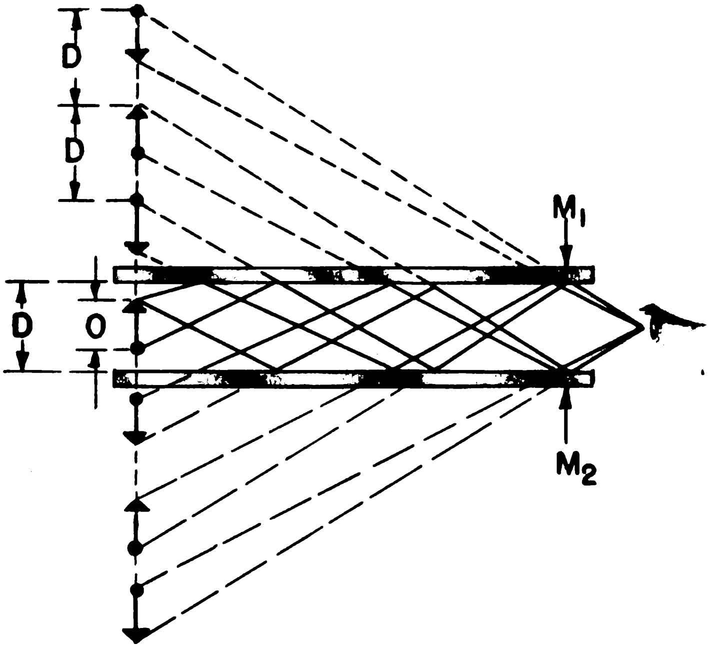


Figure 19. Reflections on two parallel plane mirrors.

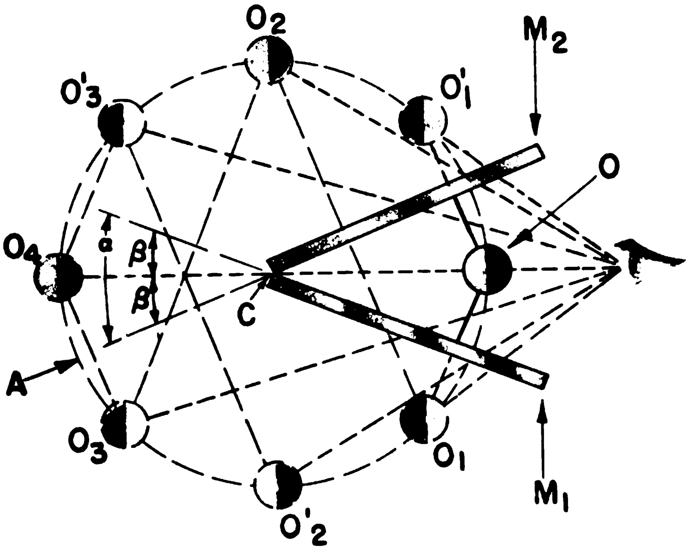


Figure 20. Reflections on two intersecting plane mirrors.

In figure 21 is shown a soldier standing in front of the large concave mirror M at some distance to the left of the center C; the height of the soldier is equal to P—p. An observer at E will see a real but inverted image of the soldier. This image will be situated between the center C and the focal point F. If the soldier increases the distance between himself and the mirror his image will move towards the focal point and, at a large distance, the image plane will coincide with the focal plane of the mirror. No matter how much more the man moves away from the mirror, the image will appear to remain fixed. If, on the other hand, the soldier occupies a position between the focal point F and the vertex of the reflecting surface, a position illustrated in figure 22, the observer will see an erect but virtual image P′—p′ behind the mirror M.

The method of determining the position and the size of the image is illustrated in figure 23. First, draw the straight line Q—q normal to the axis and through the vertex of the reflecting surface. Next, draw the straight lines P—Q and p—q, or as many lines as desired, all parallel to the axis. Draw straight lines from P and p through the center C to q and Q, respectively. A straight line from Q through the focal point F will intersect the straight line P—q at the point P′, and the straight line from q through F will intersect the iine p—Q at the point p′. The points P′ and p′ are the conjugate points to the object-points P and p, respectively; hence, we find the image P′—p′ of the rifle between the center C and focal point at a distance s′ from the reflecting surface of the mirror.

If the radius of curvature of the reflecting surface is known, and the height A of the object and the object distance s are given, we can determine the focal length f, the image distance s′, and the height A′ of the image with the aid of the following equations:

$$f = +\frac{R}{2} \tag{19}$$

$$s' = \frac{fs}{s - f} \tag{20}$$

$$A' = -\frac{As'}{s} \tag{21}$$

A negative value for A′ indicates that the image is inverted.

In figure 24 are shown four men, designated by the letters a, b, c, and d, at different distances from a large concave reflector. Each of the men is six feet tall. The men are stationed at the following distances from the mirror: a = 12.5 feet, b = 10 feet, c = 7.5 feet and d = 2.5 feet. What is the focal length of the mirror, and what are the image distances and the heights of the images when the radius of curvature of the reflecting surface is equal to − 10 feet?

We have: $A = 6$ ft. and $R = -10$ ft.

Apply equation (19) and find: $f = -5$ ft.

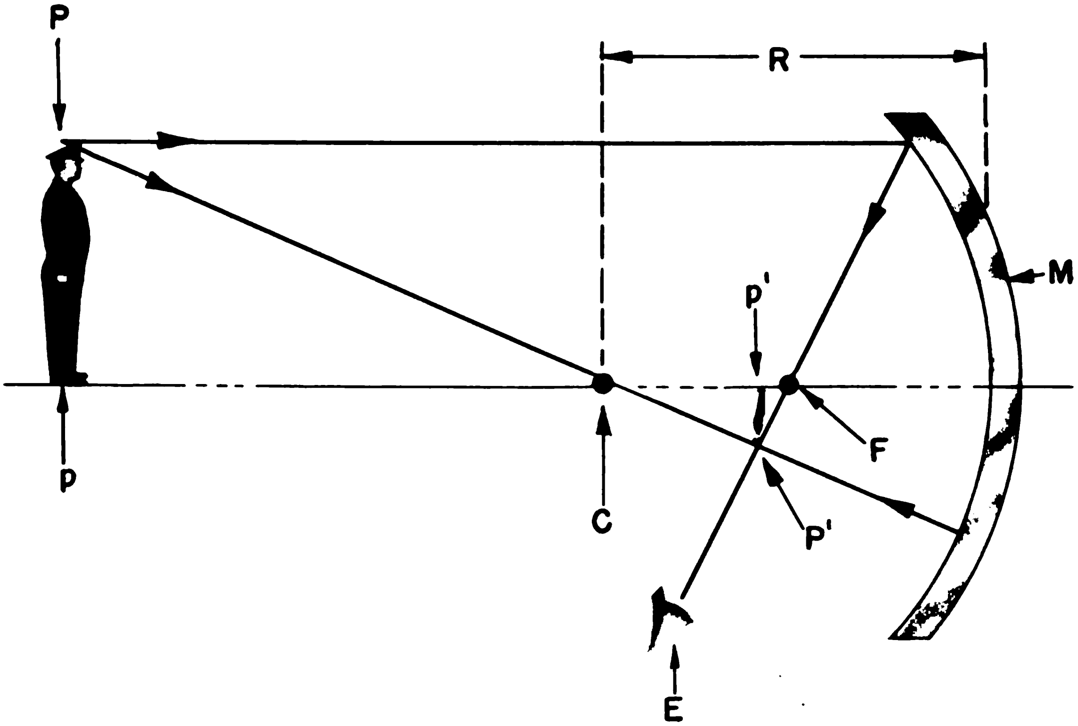


Figure 21. The concave mirror forms a real but inverted image.

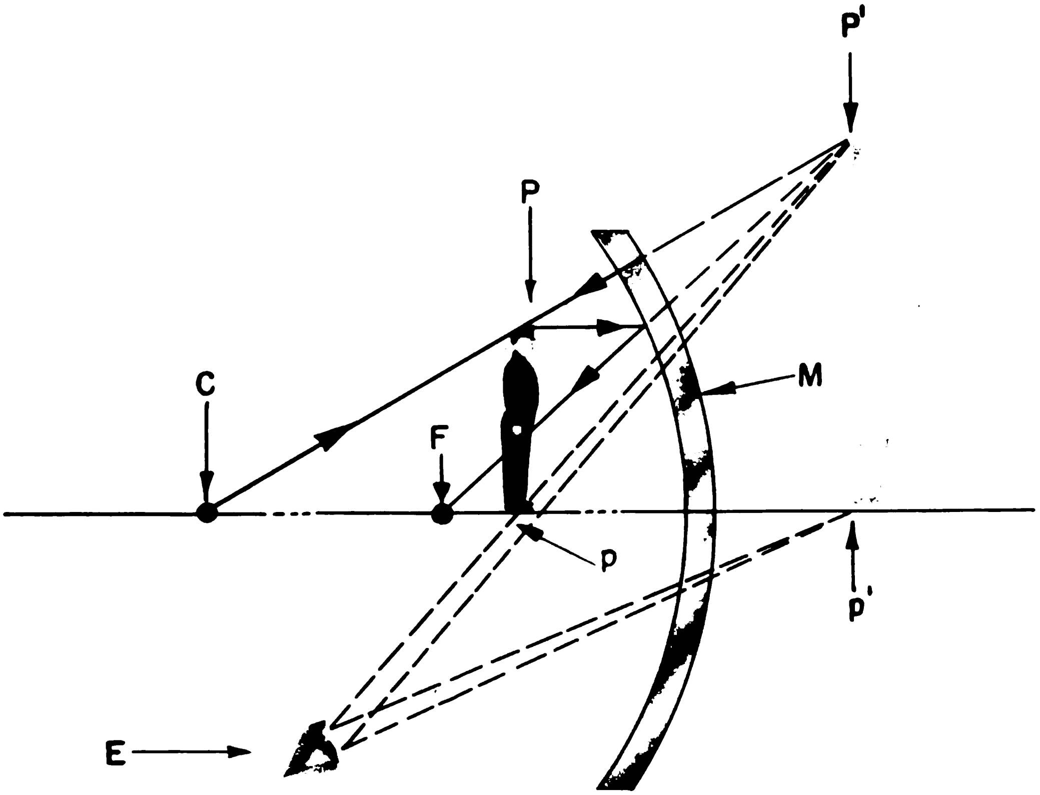


Figure 22. The concave mirror forms a virtual but erect image.

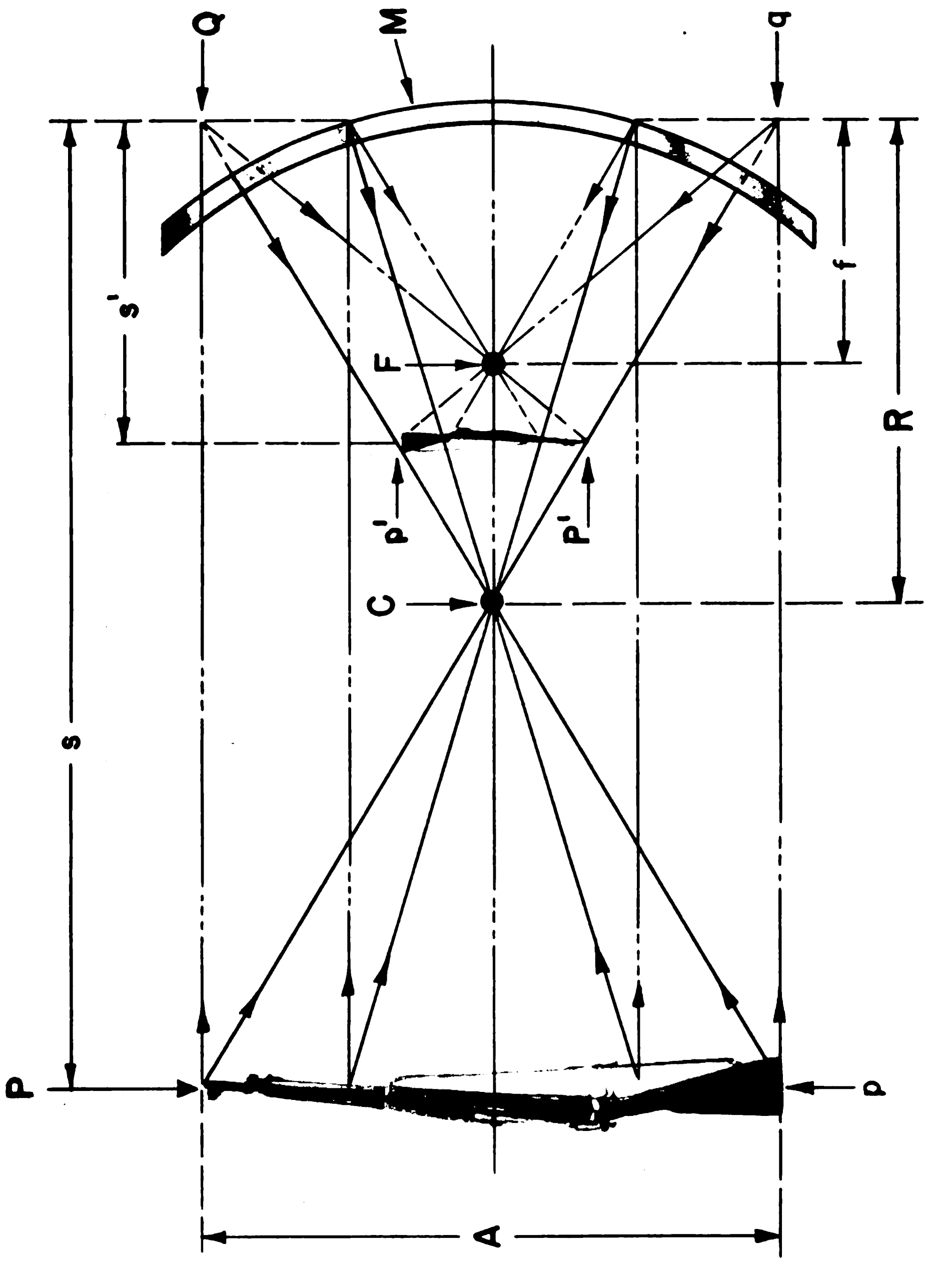


Figure 23. Constructing the image formed by a concave mirror.

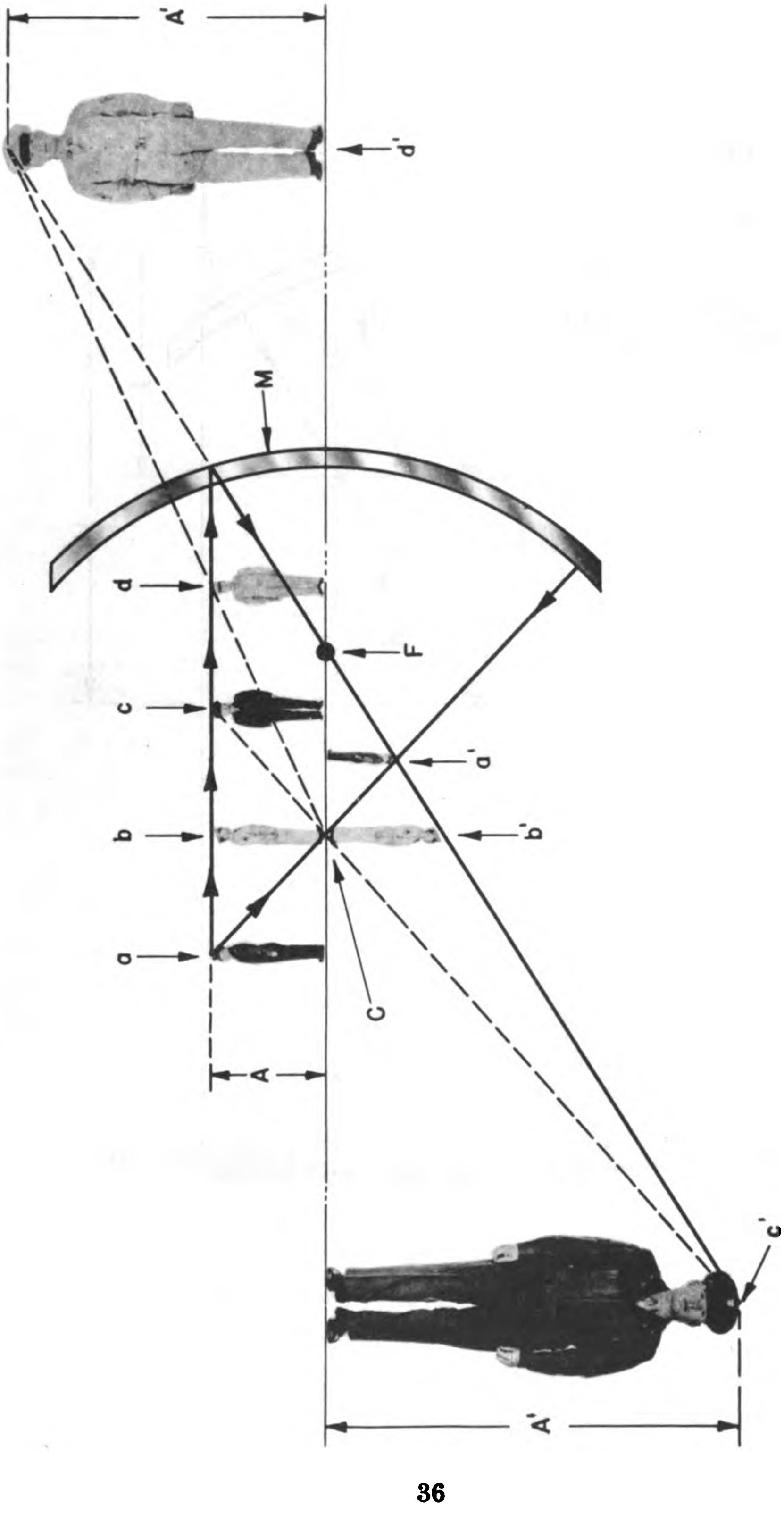


Figure 24. Conjugate points with respect to the concave mirror.

Substituting the various values of s into equations (20) and (21) we find:

s =	— 12.5 ft.	— 10 ft.	— 7.5 ft.	— 2.5 ft.
s′ =	— 8.3 ft.	— 10 ft.	— 15 ft.	+ 5 ft.
A′ =	— 4 ft.	— 6 ft.	— 12 ft.	+ 12 ft.

16. Convex Mirror

The spherical convex mirror will produce a virtual erect image of an object in front of the reflecting surface. The method employed in determining the position and the size of the image is identical with the one applied in the case of the concave mirror. In figure 25 the object shown is a rifle placed at a distance s from the mirror M, and the height of the rifle is A. We determine the position of the image and its height as follows:

Draw the straight line Q—q normal to the axis through the vertex of the reflecting surface, and then draw straight lines from points P and p, parallel to the axis. Next, draw lines from Q and q to the focal point F, and straight lines from P and p to the center C. The lines P—C and Q—F intersect at the point P′, and the lines p—C and q—F at the point p′. The points P′ and p′ are the conjugate points to the object-points P and p, respectively, and the image of the rifle is found between these points.

If the rifle is moved towards the left its image will move towards the focal point F. For infinitely distant targets, the image-plane will coincide with a plane through the focal point F.

In figure 26 are shown three objects in front of the convex mirror M. Each object is 1.5 feet high. The candle (a) is 3 feet from the mirror, the lamp (b) 2 feet, and the desk-lamp (c) one foot. The radius of curvature of the reflecting surface is + 2.5 feet. What are the values for f, s′, and A′?

We have:

$$A = 1.5 \text{ ft} \qquad \text{and} \qquad R = +2.5 \text{ ft}$$

Substituting these values and the various values of s into equations (19), (20), and (21) we obtain the following results:

$f = 1.25$ ft.

s =	— 3.00 ft.	— 2.00 ft.	— 1.00 ft.
s′ =	+ 0.88 ft.	+ 0.77 ft.	+ 0.56 ft.
A′ =	+ 0.44 ft.	+ 0.58 ft.	+ 0.84 ft.

17. Spherical Aberration

Equation (20) has been derived from the law of the conjugate foci which is mathematically expressed as follows:

$$\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} \qquad (22)$$

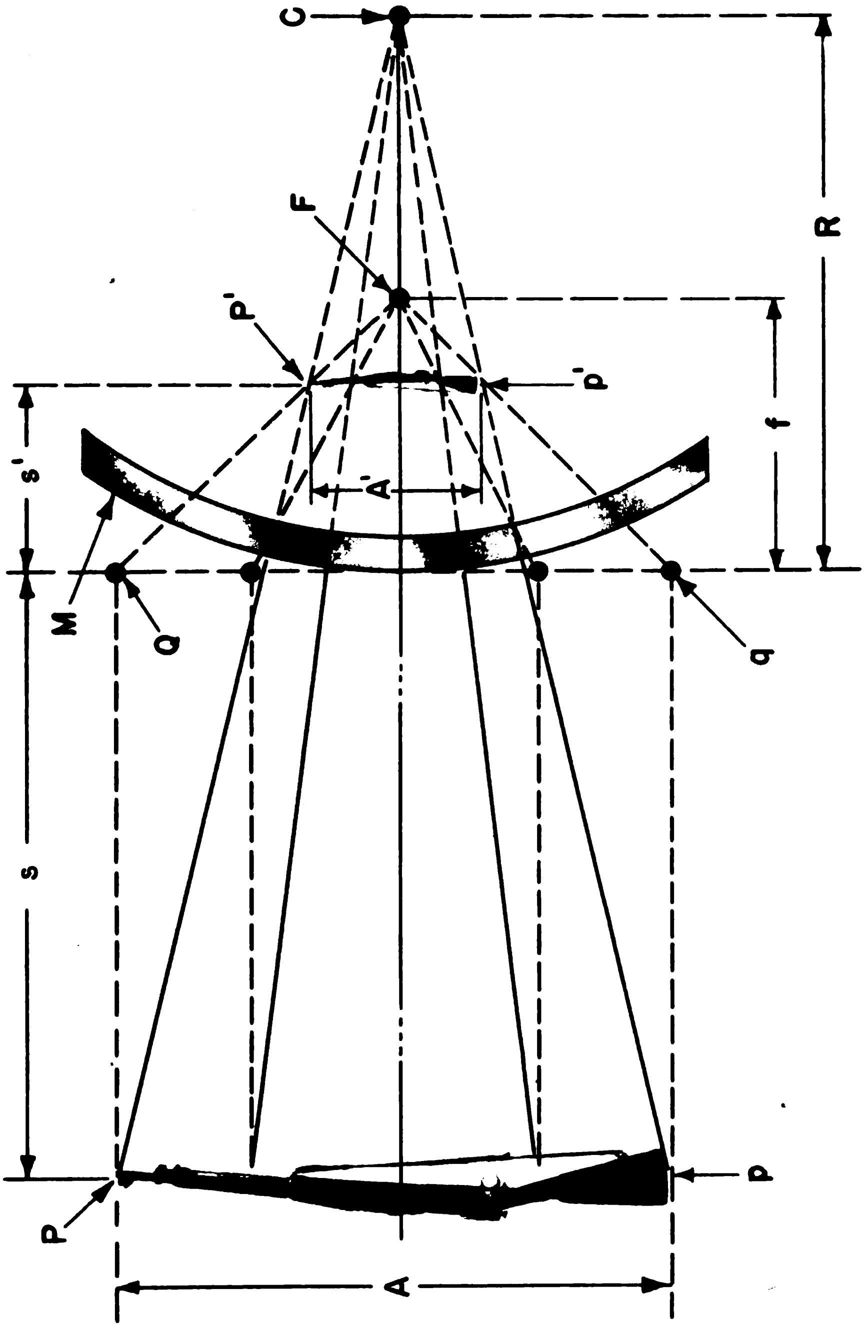


Figure 25. Constructing the image formed by a convex mirror.

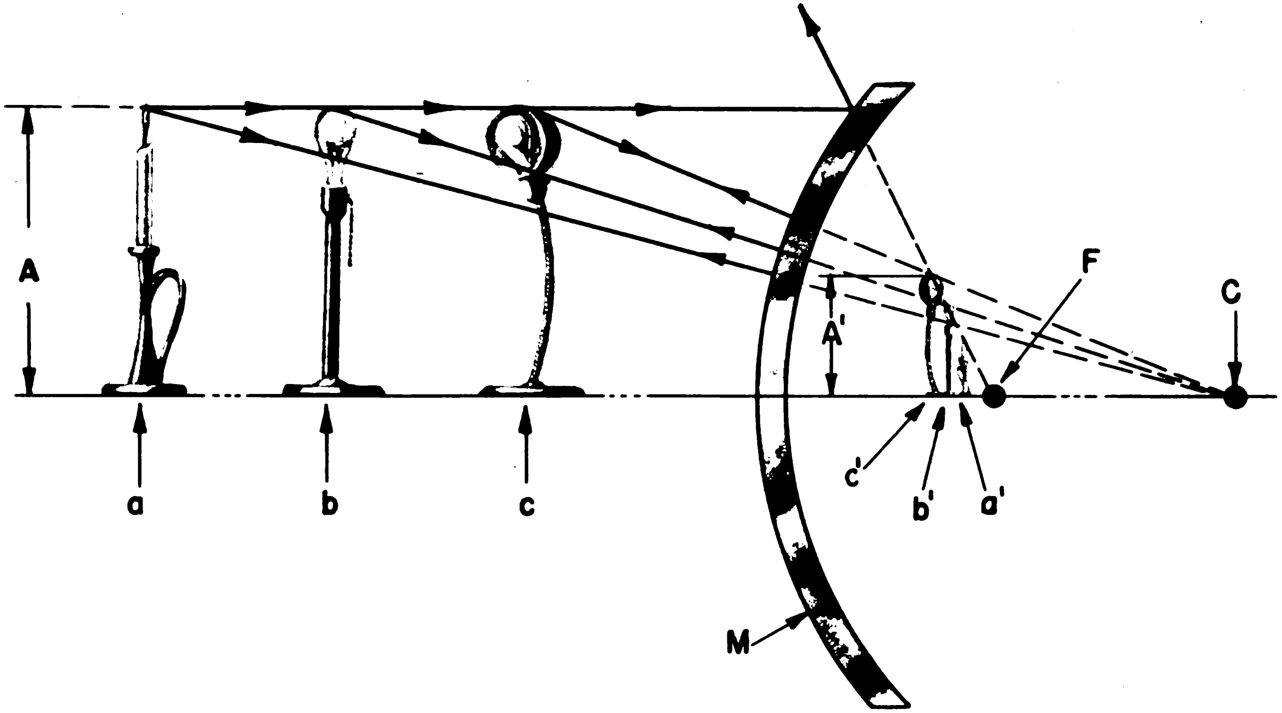


Figure 26. Conjugate points with respect to the convex mirror.

If the object is at an infinite distance $s = \infty$, and the above equation is reduced to:

$$s' = f \tag{23}$$

However, these conditions are fulfilled only as long as we consider rays of light traveling close to the axis. As soon as we increase the aperture 2h of the mirror (fig. 27), the rays reflected near the edge of the mirror will intersect the axis at the point F′ located between the focal point F and the vertex of the mirror. This deviation, that is, the distance F—F′, is known as the *"LONGITUDINAL SPHERICAL ABERRATION."*

The amount of spherical aberration produced by a spherical concave mirror depends on the focal length of the mirror and the aperture of the same. The two zero rays marked by the letter z in figure 27 are very close to the axis. After reflection by the mirror they intersect the axis at the focal point F, thus satisfying the condition expressed in equation (23). The two marginal rays m are separated from the axis by the distance h. At the point of incidence the rays will include the angle α with the normal to the reflecting surface. The reflected rays will intersect the axis at the point F′ and, proceeding, they will intersect the focal plane at the points S and S′, respectively. The distance between these points is termed the *"LATERAL SPHERICAL ABERRATION."* The magnitude of these errors can be computed as follows:

$$\sin \alpha = \frac{h}{2f} \tag{24}$$

$$\text{S.A.} = F - F' = \left\{ \frac{1}{\cos \alpha} - 1 \right\} f \tag{25}$$

$$\text{L.S.A.} = s - s' = 2\ (\text{S.A.})\ \tan 2\alpha \tag{26}$$

Let us assume, for example, that the mirror shown in figure 27 has a focal length of f = 0.75 inches, and that the marginal rays are reflected at the heights (a) 0.375 inches and (b) 0.75 inches. What are the errors? Substituting the values of f and h into the above equations we find:

	(a)	(b)
h =	0.375	0.75
α =	14°28′39″	30°
S.A. =	0.025	0.116
L.S.A. =	0.028	0.402

18. Parabolic Mirror

The parabolic mirror has one great advantage over the spherical reflector since it will produce an image of distant objects free from spherical aberration, that is, the marginal rays, as well as the zero rays, will be reflected so that they all intersect the axis at the point F. In order to determine the curvature of the reflecting surface of a parabolic mirror whose focal length is f, proceed as follows: Lay off the distance f on the x-axis, to the left of the vertex V (fig. 28). Divide the distance F—V in as many parts x_v as desired and draw a straight line, normal to the x-axis, through each point x.

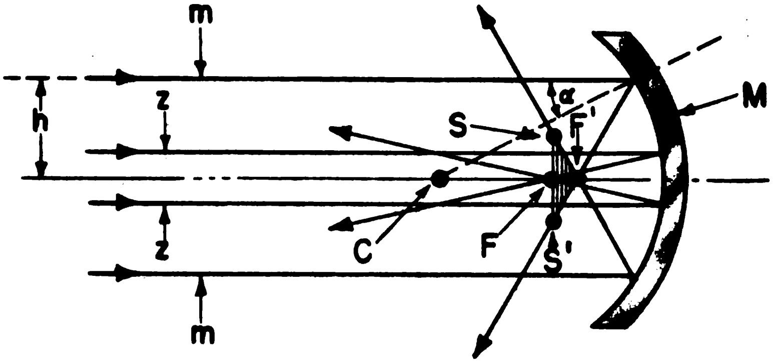


Figure 27. Spherical aberration introduced by a spherical mirror.

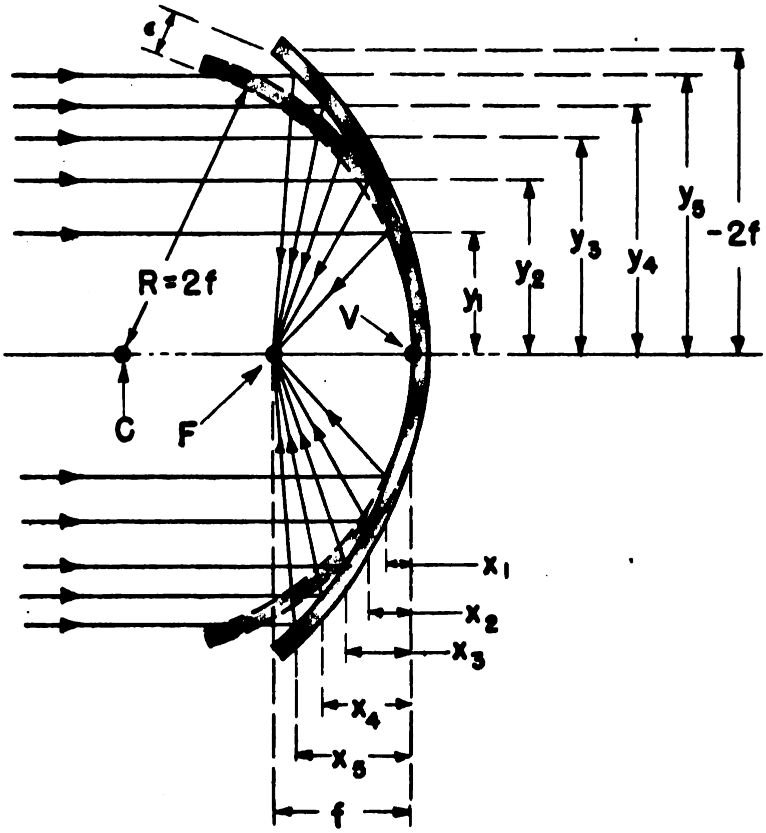


Figure 28. The parabolic mirror.

On these vertical lines lay off points separated from the x-axis by the distance y_v, one point above and another below the axis as shown in figure 28. The magnitude of the distance y_v is computed thus:

$$y_V = \quad 2\sqrt{fx_V} \tag{27}$$

Connecting all the points computed we obtain the outlines of the reflecting surface of the parabolic mirror. The radii of curvature in dash-lines refer to a spherical surface whose focal length is equal to that of the parabolic mirror. The vertex of the spherical surface coincides with that of the parabolic reflecting surface.

A glance at the diagram in figure 28 will reveal the fact that the shape of the parabolic surface deviates constantly from that of the spherical mirror, the more so, the greater the relative aperture of the mirror. The magnitude ε of this deviation at any point can be computed with the following equation:

$$\varepsilon = 2f\left\{\frac{y^4}{128f^4} + \frac{3y^6}{3072f^6} - \frac{5y^8}{3276f^8} + \dots\right\} \tag{28}$$

Let us construct a parabolic mirror whose focal length is equal to — 1.20 inches. We lay off the distance F—V into six equal parts, each two points separated by a distance of 0.20 inches. Now draw a straight line (normal to the x-axis) through each of these points. Substitute the x values into equation (27) and find the corresponding values for y. Lay these values off on the vertical line and connect these points, and we have the parabolic surface shown in figure 28. Below are tabulated the values of x_V and y_V.

V =	1	2	3	4	5	6
$x_V =$	— 0.2	— 0.40	— 0.60	— 0.80	— 1.00	— 1.20
$y_V =$	± 0.98	± 1.39	± 1.70	± 1.96	± 2.19	± 2.40

Substitute the values $y_1 = 0.98$ and $y_6 = 2.40$ into equation (28) and find: $\varepsilon_1 = 0.009$ and $\varepsilon_6 = 0.356$.

It will require special grinding machines to remove a layer of glass 0.356 inches thick from the margin of the mirror. However, in the case of the large reflectors used in astronomical instruments, the values of ε are very small. For example, the parabolic reflector made by Lord Ross has a focal length of 54 feet and an aperture of 3 feet. If we substitute these values into equations (27) and (28) we find the deviation from sphericity is exceedingly small, namely, $\varepsilon = 0.0024$ millimeters. Hence, in the case of large reflectors the lens grinder will grind first a spherical surface and then remove the small mass of glass by hand.

It has been shown that all rays of light traveling parallel to the axis before incidence at the parabolic reflecting surface will come to a common focus at F. Therefore, if we place a light source L at the focal point of the reflector, the beam of light reflected by the mirror will travel parallel to the axis as illustrated in figure 29. This type of reflector is used in automobile headlights, with searchlights, and many other illuminating devices.

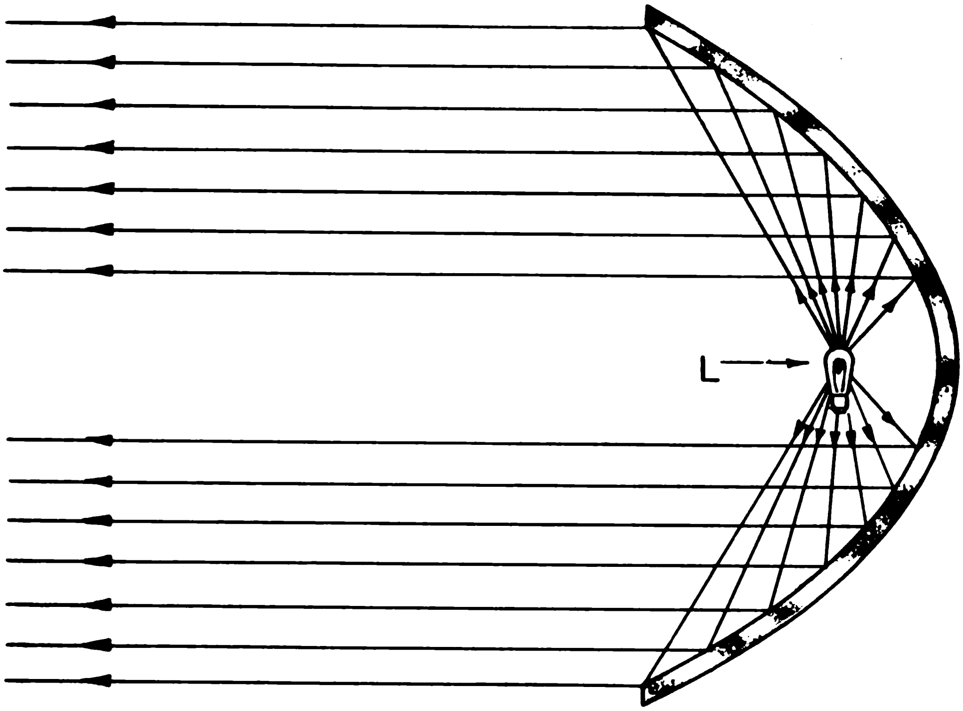


Figure 29. The parabolic reflector.

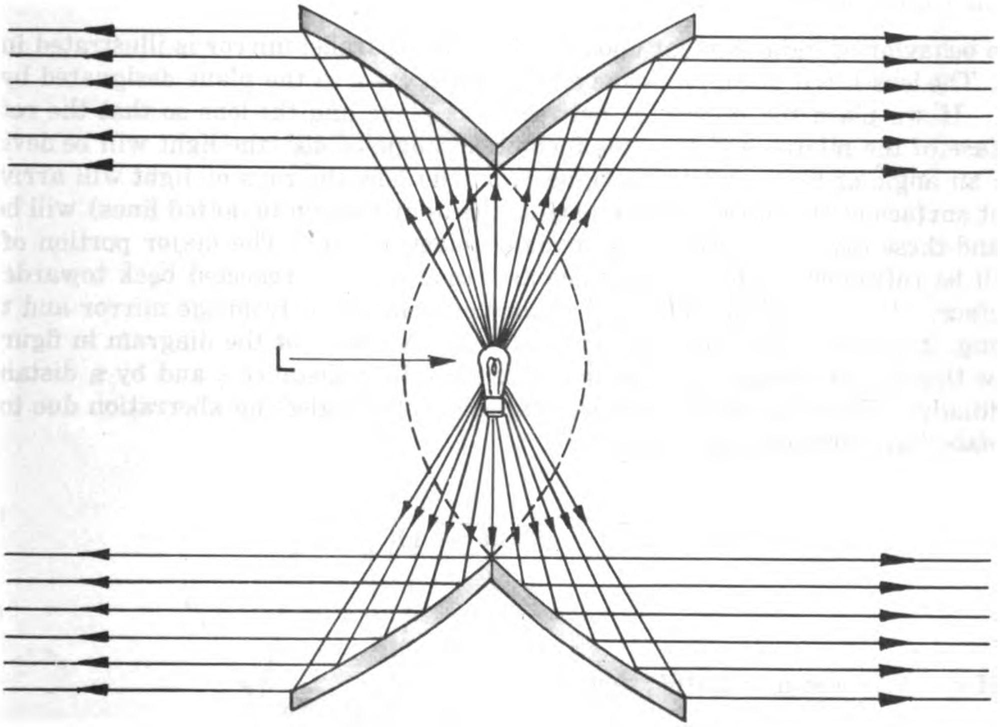


Figure 30. The double-parabolic reflector.

Certain types of instruments used for signaling purposes employ a double-parabolic reflector. Such a reflector is shown in figure 30. This double reflector is made up of two single parabolic reflectors; the portions near their vertices are ground off to a plane through the focal point F. The remaining parts are mounted so that their respective focal points will coincide, and at this focal point is placed the light-source L.

19. Thick Mirrors

Thick mirrors silvered on the rear surface are rarely used in the present era since perfect and durable reflecting surfaces are obtained by aluminizing the front surfaces. However, a number of instruments still contain thick mirrors in which the rear surface reflects the light. For this reason, the theory of the thick mirror will be explained for a few cases. The following abbreviations will be used in the ensuing paragraphs:

d = the axial thickness of the mirror.

n = the refractive index of the glass.

R_1 = the radius of curvature of the first surface.

R_2 = the radius of curvature of the second (reflecting) surface.

s_o' = the distance from the front surface of the mirror to the secondary image, that is, the image formed by light reflected at the front surface of the mirror.

s_8' = the distance from the front surface of the mirror to the primary image, that is, the image formed by light reflected by the silvered surface.

20. Thick Plane-Parallel Mirror

The behavior of light incident upon a thick plane-parallel mirror is illustrated in figure 31. The lens L will form an image of the object P—p in the plane designated by the letter I. If we place the mirror M between this plane and the lens so that the reflecting surface of the mirror and the axis include an angle of 45°, the light will be deviated through an angle of 90°. After emerging from the lens the rays of light will arrive at the front surface of the mirror where part of the light (shown in dotted lines) will be reflected and these rays will form the secondary image P′—p′. The major portion of the light will be refracted at the first surface and then will be reflected back towards the first surface. Here it will be refracted again upon emerging from the mirror and then, proceeding, it will form the primary image P—p. A glance at the diagram in figure 31 will show that the two images are displaced laterally by a distance η and by a distance μ longitudinally. The magnitudes of these displacements, neglecting aberration due to the "*thick plate,*" are computed as follows:

$$\eta = -\frac{2d}{\sqrt{2(2n^2-1)}} \tag{29}$$

$$\mu = \frac{2d}{n} + \eta \tag{30}$$

Hence, if d = 0.50 and n = 1.5165, then,

$$\eta = -0.373 \text{ inch} \quad \text{and} \quad \mu = +0.286 \text{ inch}$$

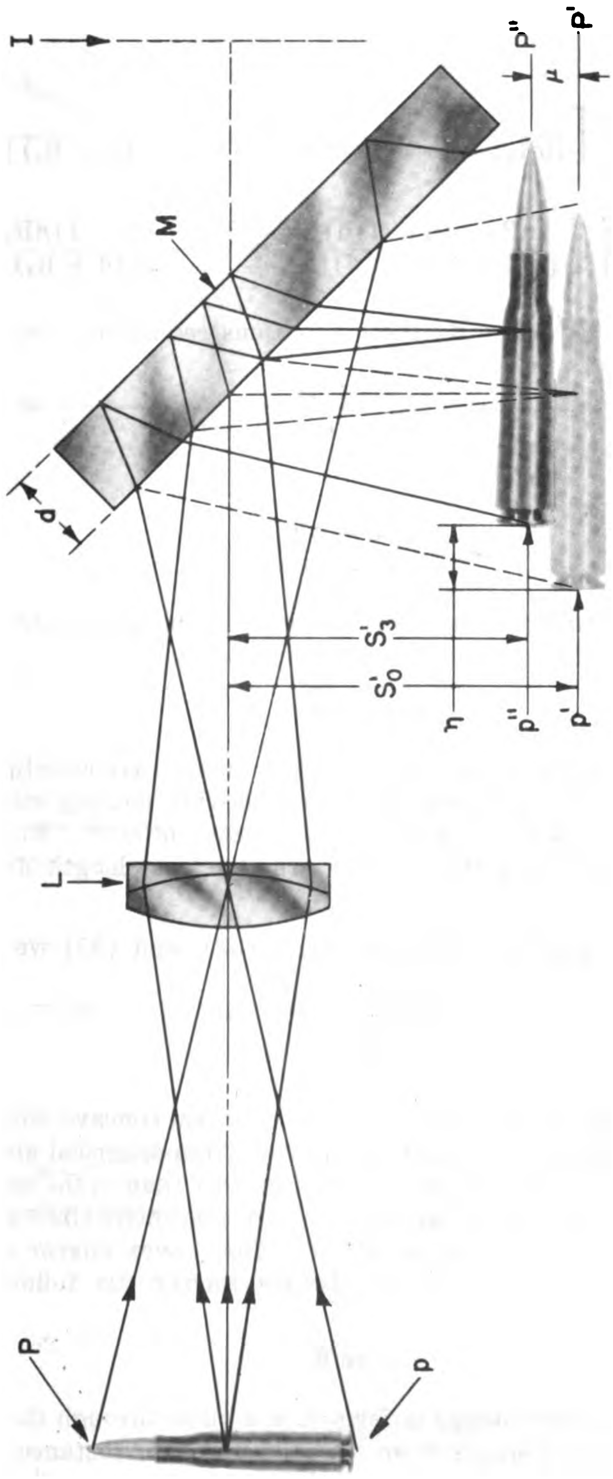


Figure 31. Reflections and refractions on a thick mirror.

21. Thick Spherical Concave and Convex Mirrors

The focal length f and the back focal length s_3' of a thick spherical mirror are computed as follows:

$$f = \left[\frac{R_1}{2}\right] \left[\frac{nR_1R_2}{[nR_1 - (n-1)d][nR_1 - (n-1)(d+R_2)]}\right] \tag{31}$$

$$s_3' = \left[\frac{R_1}{2}\right] \left[\frac{[nR_1 - (n-1)d][2d+R_2] - (n-1)dR_2}{[nR_1 - (n-1)d][nR_1 - (n-1)(d+R_2)]}\right] \tag{32}$$

In the special case when $R_1 = d + R_2$, these equations reduce to:

$$f = \frac{nR_1R_2}{2[nR_1 - (n-1)\,d]} \tag{33}$$

and

$$s_3' = \frac{(2d + nR_2)R_1}{2(d + nR_2)} \tag{34}$$

The distance s_o' from the front surface of the mirror to the secondary image is equal

$$s_o' = \frac{R_1}{2} \tag{35}$$

The positions of the primary and the secondary images are clearly illustrated in figure 32 in the case of a concave mirror and in figure 33 when dealing with convex mirrors. Assume the following constants are given for a concave mirror: $R_1 = -2$ inch, $R_2 = -3$ inch, $d = 1.0$ inch, and $n = 1.5165$. What are the focal length and the image distances for this mirror?

Substituting these values into equations (31), (32), and (35) we find:

$f = -1.281$ inch $s_o' = -1$ inch $s_3' = -0.718$ inch

22. Mangin Mirror

The French colonel, A. Mangin, constructed a spherical concave mirror in 1875 which produced an image of distant objects practically free from spherical aberration. On the other hand, the chromatic aberration is somewhat greater than in the case of the ordinary type of back surface mirror in which the surfaces are concentric since a greater marginal thickness is required as shown in figure 34(a). The convex mirror requires a greater axial thickness (see fig. 34(b)). In the Mangin mirror the following condition is satisfied:

$$s_3' = R_1 \tag{36}$$

or, in other words, the primary image is formed in a plane through the center C_1. If the object is situated at a finite distance from the mirror as, for instance, the soldier in figure 34(a), the primary image will be located between the centers C_1 and C_2. The sec-

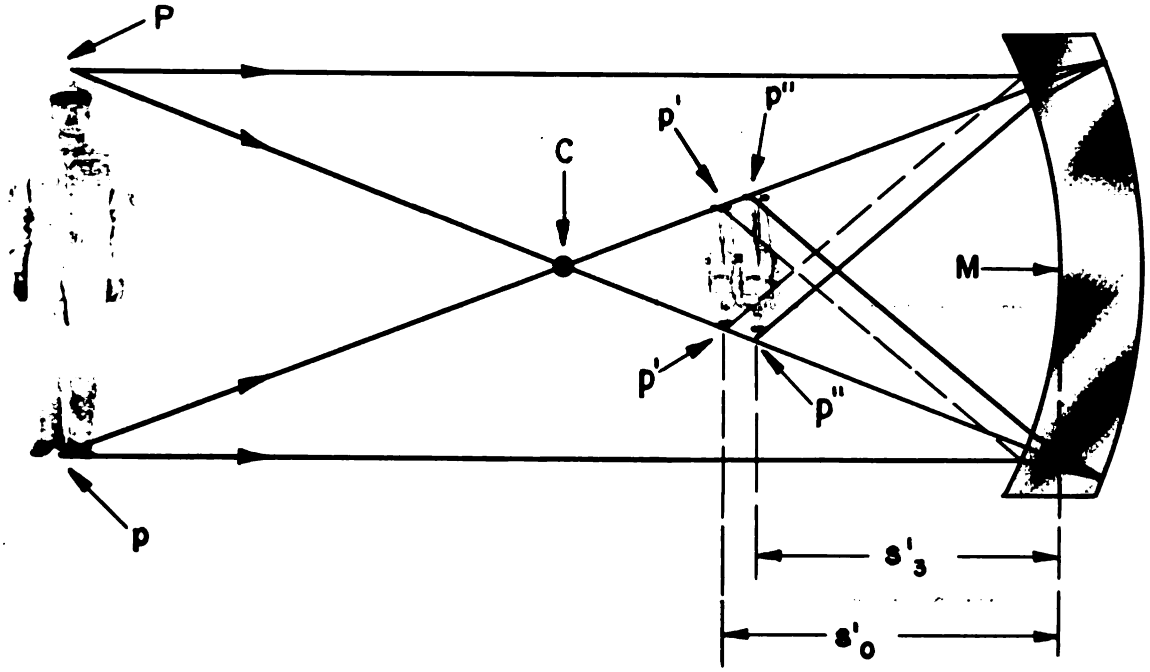


Figure 32. Reflections and refractions on a thick concave mirror.

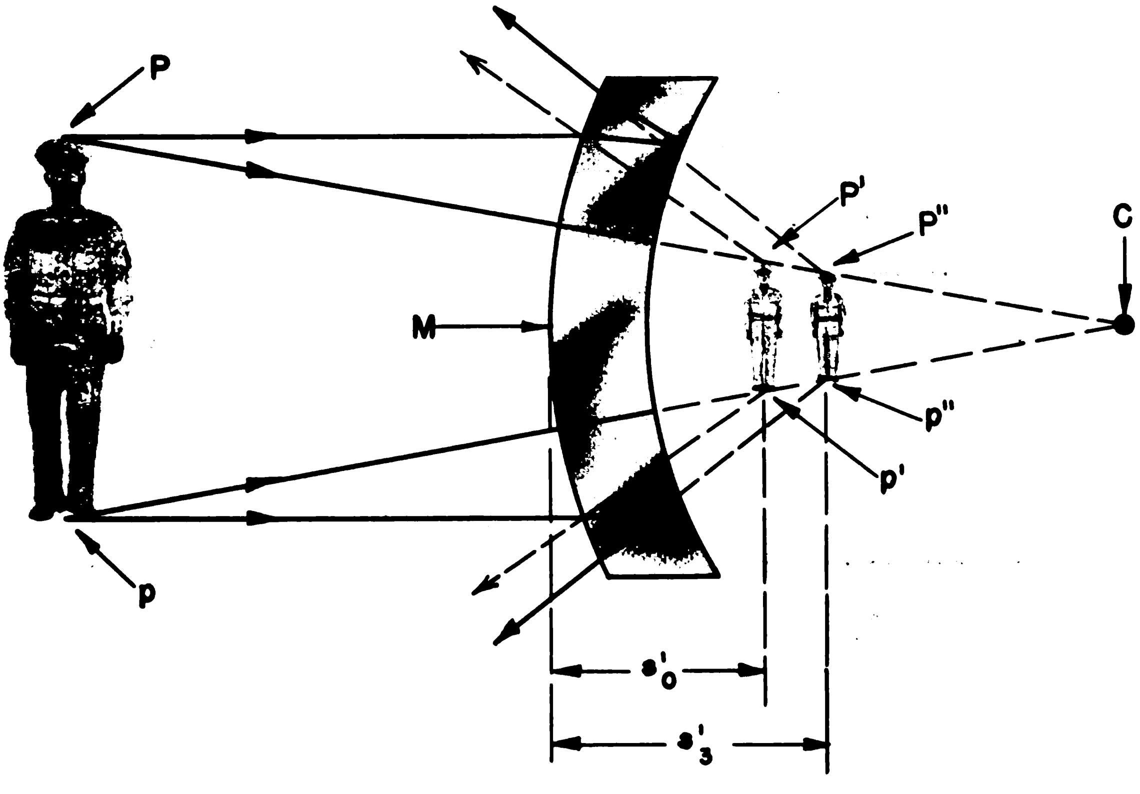


Figure 33. Reflections and refractions on a thick convex mirror.

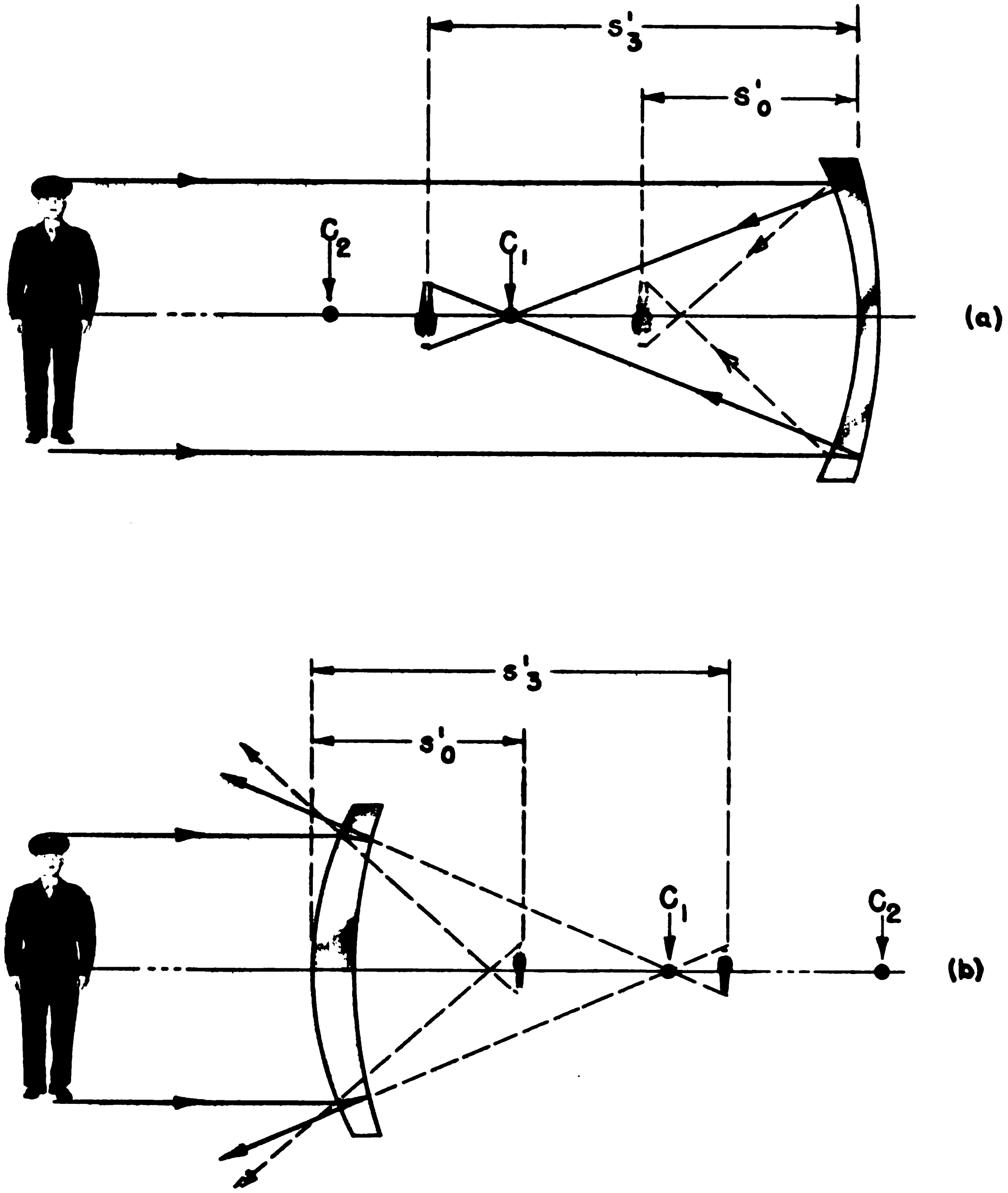


Figure 34. The Mangin mirror.

ondary image will be found nearly half-way between C_1 and the mirror. A glance at the diagrams in figure 34 will reveal the fact that the distance between the primary and secondary images is greater than in the case of the concentric mirror and, consequently, the primary image will appear clearer and better defined if produced by a Mangin mirror. The focal length of this type of mirror is computed as follows:

$$f = R_1 - \frac{dR_1}{nR_1 - (n-1)d} \tag{37}$$

In order to determine the constants of a Mangin mirror we select, arbitrarily, values for d and R_1 and then compute the value of R_2 as follows:

$$R_2 = \frac{2\,[nR_1^2 - (2n-1)dR_1 + (n-1)d^2]}{(2n-1)R_1 - 2(n-1)d} \tag{38}$$

The equations just given hold good for concave as well as for convex mirrors. For a numerical example, let us assume values as given:

$R_1 = -2$ inch, $d = 0.10$ inch, and $n = 1.5165$

Applying successively equations (38) and (37) we find:

$R_2 = -3.107$ and $f = -2.065$

Tracing a marginal ray ($h_1 = 1$ inch) through the mirror we find that the spherical aberration is equal to 0.015 inch, which is considerably less than that for the concentric mirror.

23. Straubel Mirror

The Straubel mirror is used in searchlights and is designed so that the primary and the secondary images lie in one and the same plane. Hence, the following condition is fulfilled:

$$s_0' = s_3' = \frac{R_1}{2} \tag{39}$$

and the focal length of the mirror is computed thus:

$$f = \frac{R_1[nR_1 - (n+1)d]}{2[nR_1 - (n-1)d]} \tag{40}$$

In the designing of a Straubel mirror we proceed as in the case of the Mangin mirror, that is, we assume, arbitrarily, values for d, n, and R_1 and then compute the magnitude of the reflecting surface with the aid of the following equation:

$$R_2 = \frac{n^2(R_1 - d)^2 - d^2}{n^2R_1 - (n^2-1)d} \tag{41}$$

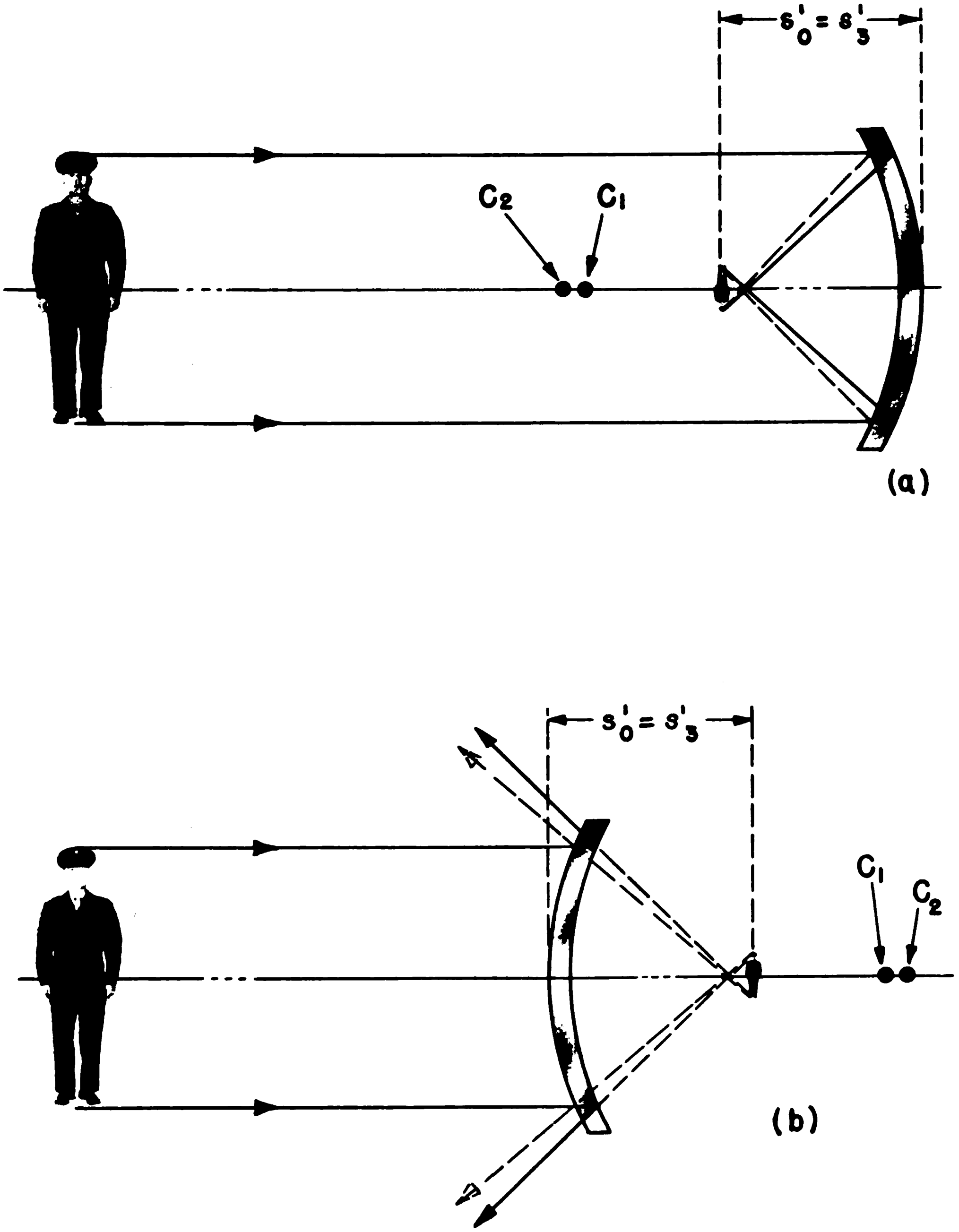


Figure 35. The Straubel mirror.

Again, we assume the following values as given:

$R_1 = -2$ inch $\qquad d = 0.10$ inch, and $n = 1.5165$

Applying successively equations (40) and (41) we find:

$f = -1.065$ inch $\qquad R_2 = -2.142$ inch

During the first World War there were manufactured by Carl Zeiss, Jena, reflectors for use with searchlights which were free from spherical aberration for extremely large aperture ratios. These mirrors, also designed by Dr. Straubel, have a parabolic front surface and a silvered deformed parabolic rear surface.

SECTION III

PLANE-PARALLEL PLATE

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24. Explanations

If a plane-parallel plate of sufficient diameter is placed in parallel light in front of a telescope objective it will not affect the constants of the instrument nor the image quality. However, it will reduce the light transmission to a slight degree. On the other hand, if the plate is placed in convergent or divergent light, say between the objective and the reticle, it will introduce chromatic and spherical aberration in the image, and it will also become necessary to displace the reticle.

The majority of fire-control instruments employ prisms or prism-systems for the purpose of erecting the image formed by the objective. As a rule, the prisms are placed between the objective and the reticle, and this position of the prisms will introduce the same errors, and of the same magnitude, as a plane-parallel plate whose thickness is equal to the length of the light path in the prisms, assuming that the prisms and the plate are made of the same glass. Hence, for convenience, when computing a telescopic system, the designer will consider the prisms replaced by a plane-parallel plate of equal light path.

25. Longitudinal Displacement of Image-Plane

The rays of light passing through the objective O (see fig. 36(a)) will form an inverted image of a distant target in the image plane i which, in this case, coincides with the focal plane of the objective. The equivalent focal length of the objective is designated by the letter f_o. If we now place the plane-parallel plate Pl at a distance A behind the objective, the image will then be formed in the plane I, that is, at the distance Δ behind the plane i. The magnitude of this displacement is computed as follows:

$$\Delta = \frac{(n-1)d}{n} \tag{42}$$

where d is the thickness of the plate and n is the refractive index of the glass. B, the distance from the plate to the image plane I, is determined as follows:

$$B = f_o + \Delta - (A + d) = f_o - \left(A + \frac{d}{n}\right) \tag{43}$$

Example: The following constants are known:

$A = 1.70$ inch $\quad d = 3.90$ inch $\quad f_o = 6.00$ inch and $n = 1.5165$

Substituting the values of d and n into equation (43) we find $\frac{d}{n} = 2.572$ and $B = 1.728$.

In figure 36(b) the plate Pl has been replaced by the Abbe inverting prism Pr. The prism is made of the same glass used in the plate and the length of the light path is equal to the thickness of the prism, namely, $d = 3.90$ inch. Hence, if $A = 1.70$ inch again, B will also be equal to 1.728 inch.

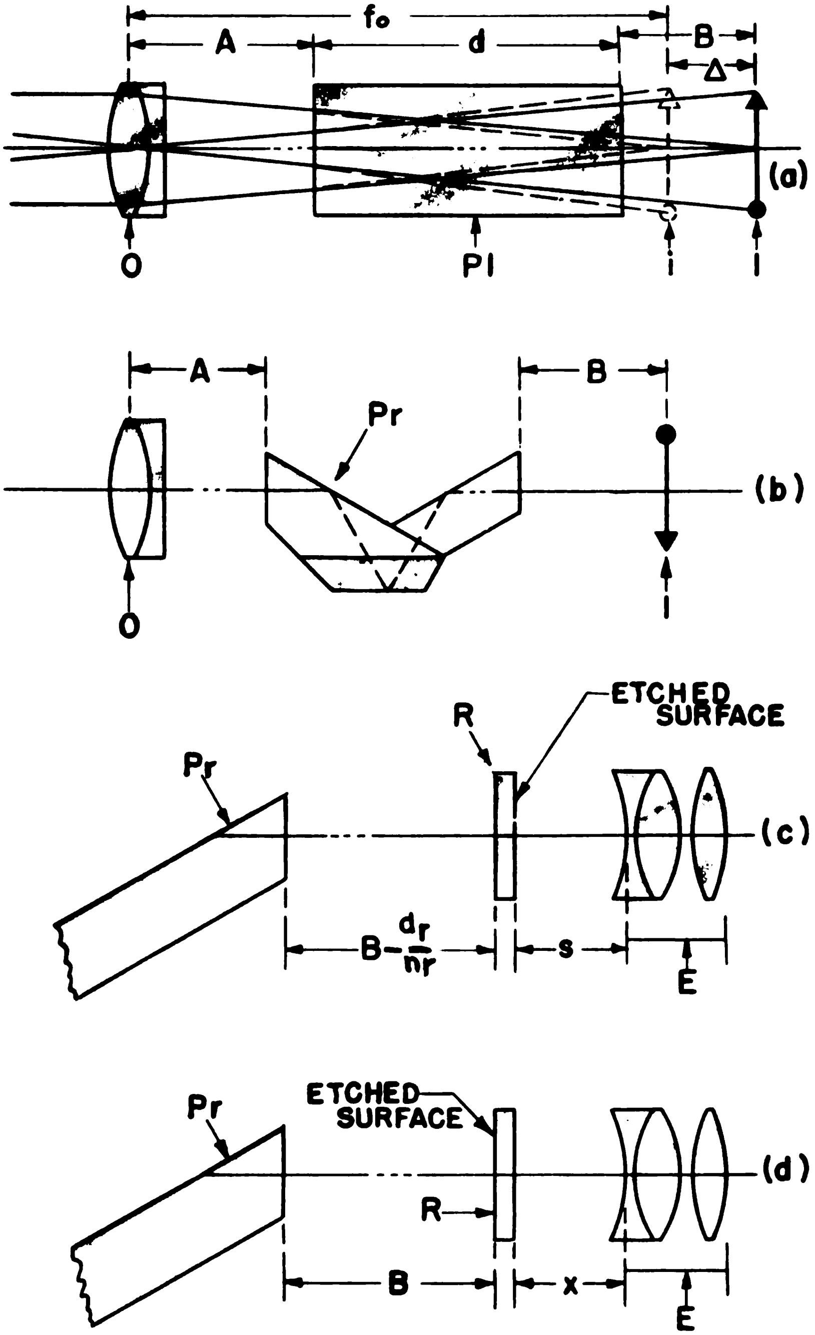


Figure 36. The displacement of the image plane.

Reticles are plane-parallel plates and their introduction into an optical system will also require a slight displacement. In figure 36(c) the rear surface of the reticle R is etched, while in figure 36(d) the front surface of the reticle is etched. Pr represents the rear end of the Abbe prism and E the eyepiece. Let the front focal length of the eyepiece be designated by the letter s. Let us assume that $s = 0.642$ inch, the thickness of the reticle is $d_r = 0.12$ inch and its refractive index is $n_r = 1.5838$, then,

$$\frac{d_r}{n_r} = 0.076 \text{ inch}$$

In the case when the rear surface of the reticle is etched, the distance from the rear surface of the prism Pr to the front surface of the reticle will be equal to $B - \frac{d_r}{n_r} = 1.652$ inch, and the distance from the rear surface of the reticle R to the front surface of the eyepiece E will be equal to the front focal length of the eyepiece, that is, $s = 0.642$ inch. If, on the other hand, the front surface of the reticle is etched, then the distance between Pr and R remains equal to $B = 1.728$ inch, and the reticle and the eyepiece are separated by the distance $x = s - \frac{d_r}{n_r} = 0.566$ inch.

26. Chromatic Aberration

The objective O (fig. 37) is chromatically well corrected before the plate Pl is placed into the light-path. The paraxial ray p will, after emerging from the objective, proceed toward the focal point F on the axis. A paraxial ray travels very close to the axis, but in order to illustrate more clearly the aberration caused by the plate, the ray is shown entering the objective near the rim. Placing the plane-parallel plate (or prisms) between the objective and its focal plane, the ray p will be dispersed into a series of rays when entering the plate. Each of these rays will be of a different color. All rays will be refracted towards the normal, rays of light of the shorter wave-lengths more so than the rays of the longer wave-lengths. After another refraction at the rear surface the rays will emerge from the plate and then proceed in the original direction, parallel to each other. The red ray will intersect the axis at the point r and the blue ray at b. In telescopic systems it is customary to correct the chromatic aberration for light corresponding to the C (red) and F (blue) lines of the spectrum. The distance CA between the points r and b is the CHROMATIC ABERRATION introduced by placing the plate Pl behind the objective. From the diagram in figure 37 it may be seen that the blue ray intersects the axis at a greater distance from the plate than the red ray, that is, ${}_Fs' > {}_Cs'$, and in this case we speak of chromatic over-correction. The magnitude of this error is computed as follows:

$$CA = {}_Fs' - {}_Cs' = \frac{d \Delta n}{{}_Cn\,{}_Fn} \qquad (44)$$

where d is the thickness of the plate, ${}_Cn$ and ${}_Fn$ are the refractive indices of the glass corresponding to the C and F lines of the spectrum, and Δn is the difference between these indices, that is, $\Delta n = {}_Fn - {}_Cn$. It is evident that the greater the thickness d of the plate (or the length of the light-path in the prism) the greater will be the chromatic over-correction. Therefore, if the combination objective-prism is to be achromatic, the objective by itself must be chromatically under-corrected by the amount *"Kappa"* whose value is computed thus:

$$K = -\frac{CA}{f_o^2} \tag{45}$$

where f_o is the focal length of the objective. The introduction of this factor *"Kappa"* will be explained in the section on achromatic lenses.

In the numeral examples given in the following, the glasses used will have the constants:

$\Delta n = 0.00808$ $\quad {}_C n = 1.51402$ $\quad {}_D n = 1.5165$ $\quad {}_F n = 1.52210$

An objective has a focal length of 4.8 inches and a prism system is placed between the objective and the image plane; the length of the light-path in the prism system is equal to 4 inches. To what extent must the objective be chromatically under-corrected in order that the combination objective-prism be achromatic?

We have: $d = 4$ inches and $f_o = 4.8$ inches. With the aid of equations (44) and (43) we find:

$$CA = 0.01402 \text{ and } K = -0.00061$$

27. Spherical Aberration

The paraxial ray p and the marginal ray m, traveling parallel to the axis, enter the objective O as shown in figure 38. The objective is free from spherical aberration; hence, the rays, after emerging from the objective will proceed towards the focal point F. If the plate P1, or a prism system, is placed between the objective and its focal point F the points of intersection will be displaced from F to P′ for the paraxial ray, and from F to m′ for the marginal ray. The distance SA between the points p′ and m′ is called the spherical aberration and, since the rim ray m intersects the axis at a greater distance from the plate than the paraxial ray p, we speak of spherical over-correction. From figure 38 we derive easily, $s = f_o - A$. Then,

$${}_p s' = s - \frac{d}{n} \tag{46}$$

$${}_m s' = s - \frac{d \tan i'}{\tan i} \tag{47}$$

where i and i′ are the angles of incidence and refraction, respectively, at the first surface of the plate. Considering that:

$$\sin i' = \frac{\sin i}{n}$$

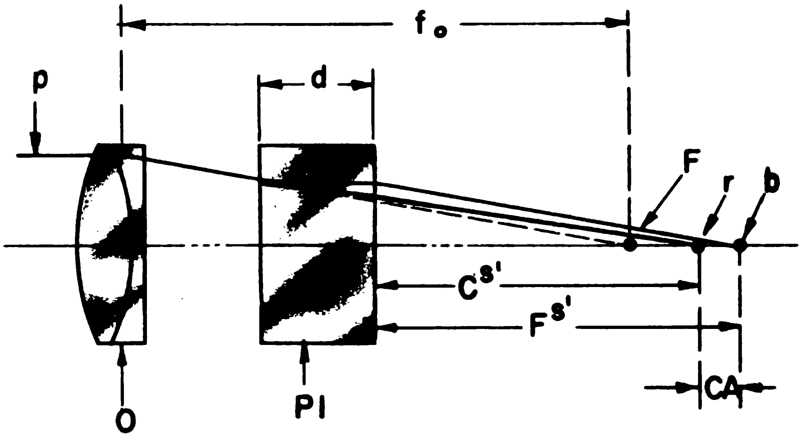


Figure 37. Chromatic aberration introduced by the plate "P1."

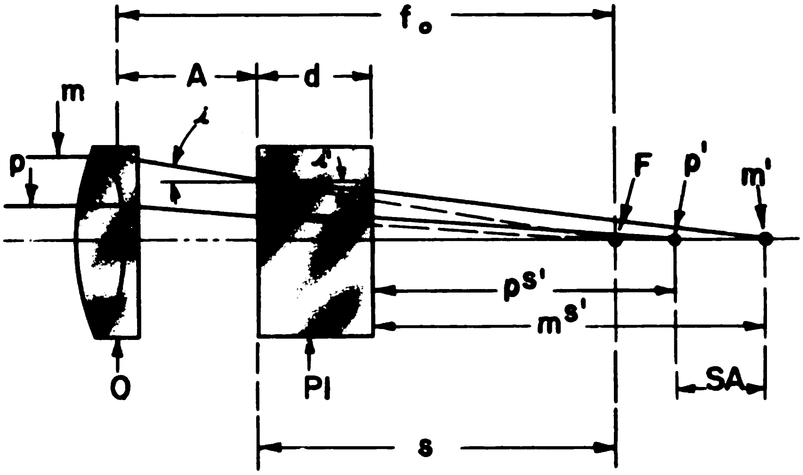


Figure 38. Spherical aberration introduced by the plate "P1."

we may write equation (47):

$$ {}_{m}s' = s - \frac{d \cos i}{\sqrt{n^2 - \sin^2 i}} \tag{48} $$

and

$$ SA = {}_{p}s' - {}_{m}s' = \frac{d}{n} \left\{ \frac{n \cos i}{\sqrt{n^2 - \sin^2 i}} - 1 \right\} \tag{49} $$

Therefore, if the combination objective-prism is to be spherically corrected, the objective by itself must be spherically under-corrected by an amount equal to

$$ \varepsilon = \frac{(1-n^2)\ d\ h_1^2}{2n^3 f_o^2} \tag{50} $$

where h_1 is the semi-aperture of the objective. Equation (50) may be derived by application of proper mathematics to equation (49). Considering the example used in the previous paragraph, to what extent must the objective be spherically under-corrected if the objective-prism combination is to be free from spherical aberration? We have $d = 4$ inch, $f_o = 4.8$ inch, $h_1 = 0.59055$ inch; hence,

$$ \varepsilon = - \frac{1.299772 \times 4 \times 0.348749}{2 \times 3.487605 \times 23.04} = - \frac{1.813179}{160.708825} = - 0.011282 $$

28. Sine Condition

If we now consider a luminous point object located off the axis, the object, in general, will not be imaged as a point due to the aberration known as coma. If, however, the *SINE CONDITION* is fulfilled for each ray coming from the object point, this aberration will disappear.

The sine condition is fulfilled when parallel beams of light, entering the objective obliquely, come to focus in the image plane of the objective. If the objective is also free from spherical aberration, the image will be free from coma. If we place a plane-parallel plate, or a prism, between the objective and its image plane, the sine condition will not be affected, but since spherical aberration has now been introduced there will be coma present in the image (see fig. 39). Therefore, if the image produced by the *"objective-prism"* combination is to be free from coma, the objective by itself must be over-corrected with regards to the sine condition by an amount ε' equal to that of the spherical under-correction; hence,

$$ \varepsilon' = - \varepsilon \tag{51} $$

29. Astigmatism and Field-Curvature

Assume the principal ray PR, including the angle α with the axis, passes through the objective (not shown in fig. 40) undeviated. After traversing the plate P1, it will pierce the image plane IP at the point a. The tangential rays TR travel very close to

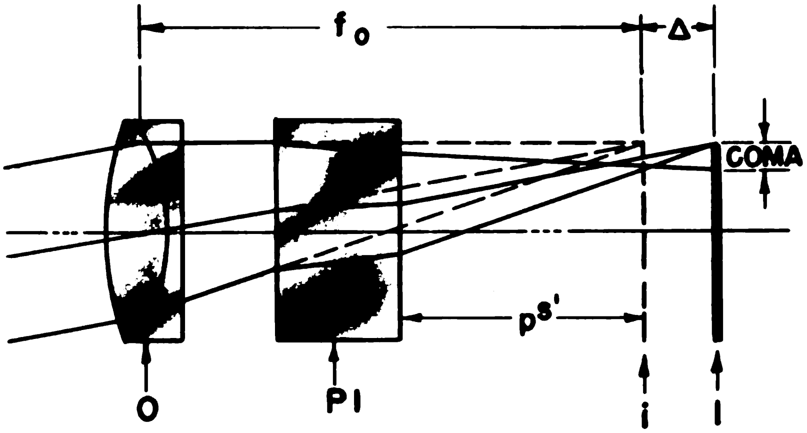


Figure 39. The coma introduced by a plate.

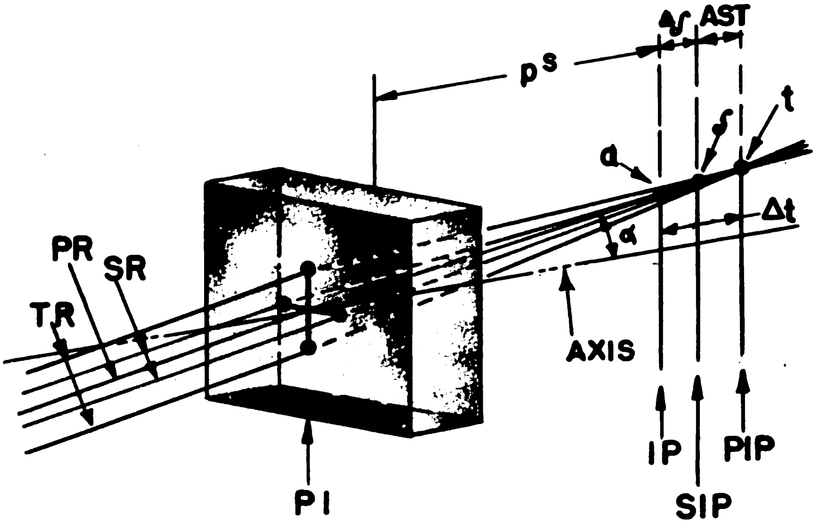


Figure 40. Astigmatism produced by a plate.

the principal ray in a plane containing the principal ray and the axis. This plane is known as the *"tangential plane."* The *"sagittal rays"* also travel very close to the principal ray but in a plane designated the *"sagittal plane,"* that is, in a plane normal to the tangential plane. If the tangential and the sagittal rays intersect the image plane also at the point a, then the combination objective-plate (or prism) will produce an image free from astigmatism and field-curvature. In the case it is said that the combination is *"anastigmatic."* However, this ideal condition is never realized in optical systems of this type. From the diagram in figure 40 it will be seen that the tangential rays intersect the principal ray at the point t which is found in the *"primary image plane"* marked PIP in the diagram. The sagittal rays will intersect the principal ray at the point S in the *"secondary image plane"* designated by the letters SIP. The distance between IP and SIP is designated by the symbol "ΔS," and the distance between IP and PIP by Δt. The magnitude of astigmatism AST and the field-curvature FC are determined as follows:

$$\mathrm{AST} = \Delta t - \Delta S \tag{52}$$

$$\mathrm{FC} = \frac{\Delta t + \Delta S}{2} \tag{53}$$

Both errors are positive; hence, the plane-parallel plate (or prism) also acts to over correct the system with regards to astigmatism and field curvature. The absolute values of these errors are computed as follows:

$$\mathrm{AST} = \frac{d \cos \alpha}{\sqrt{n^2 - \sin^2 \alpha}} \left\{ 1 - \frac{n^2 \cos^2 \alpha}{n^2 - \sin^2 \alpha} \right\} \tag{54}$$

$$\mathrm{FC} = \frac{d}{n} - \frac{d \cos \alpha}{2\sqrt{n^2 - \sin^2 \alpha}} \left\{ 1 + \frac{n^2 \cos^2 \alpha}{n^2 - \sin^2 \alpha} \right\} \tag{55}$$

where d is the thickness of the plate (or the length of the light-path in the prism), n is the refractive index of the glass, and α is the angle included by the principal ray and the axis. For example, what amounts of astigmatism and field-curvature are introduced by a plane-parallel plate 4 inches thick whose refractive index is 1.5165, if the angle included by the axis and the principal ray is 10°? We have, $d = 4.00$, $n = 1.5165$, and $\alpha = 10°$. Substituting these values into equations (54) and (55) we find:

$$\mathrm{AST} = +\,0.045 \text{ inch,} \quad \text{and } \mathrm{FC} = +\,0.0456 \text{ inches}$$

30. Distortion

The two principal rays marked P_1 and P_2 and the axis include the angles α_1 and α_2, respectively, as shown in figure 41. After passing through the objective O undeviated, the rays will intersect the image plane of the objective above the axis at the distances y_1 and y_2, respectively. If we place a plate Pl (or prism) between the objective and its focal plane, the image plane will be displaced along the axis by the distance Δ. In this position this plane will be pierced by the principal rays at the distances y_1' and y_2' above the axis. If the distances y_1' and y_2' are greater than the distances y_1 and y_2, respectively, the image will show positive, or *"cushion-shape"* distortion, that is, straight lines extending across the target will appear curved convex toward the center of the image. In the case of negative, or "*barrel-shape*" distortion the distances y_1' and y_2' are smaller than y_1 and y_2, respectively, and straight lines in the target will appear curved concave toward the center of the image.

A glance at figure 41 will reveal the fact that a plane-parallel plate (or prism) will introduce barrel-shape distortion. From the diagram we derive easily:

$$y = f_o \tan \alpha \tag{56}$$

$$y' = (f_o - \frac{d}{n}) \tan \alpha + d (\tan i') \tag{57}$$

where i′ is the angle of refraction at the first surface of the plate. The magnitude of the distortion introduced by the plate is:

$$\text{DIST} = y' - y \tag{58}$$

This equation, after several substitutions, may be written as follows:

$$\text{DIST} = - d (\sin \alpha) \left\{ \frac{1}{n \cos \alpha} - \frac{1}{\sqrt{n^2 - \sin^2 \alpha}} \right\} \tag{59}$$

Example: A plane-parallel plate, 4 inches thick, is placed between the objective and its focal plane. What is the amount of distortion introduced when the principal rays include the angles of 10° and 20°, respectively?

We have, $d = 4$ inch, $n = 1.5165$ and $\alpha = -10°$ and $-20°$.

Substitute the values into equation (59) we find,

DIST $= -0.004$ inch when $\alpha = -10°$, and

DIST $= -0.034$ inch when $\alpha = -20°$.

The minus sign indicates that we have a case of negative, or barrel-shape distortion.

31. Tilting Plane-Parallel Plate

The principle of the tilting plane-parallel plate is used in the correction of halving errors in some types of range finders. This principle will be readily understood by a glance at the diagram in figure 42. The axis will pass through the objective O and the plate P1 undeviated, and will pierce the image plane at the point a. The principal ray P, including the angle α with the axis, will also pass undeviated through the objective and, after passing through the plate, it will pierce the image plane at the point b. In both cases it is assumed that the plate is placed normally to the axis as indicated by the dotted outlines of the plate in the diagram.

If we tilt the plate about the point C on the axis through an angle δ, the axial image point will move from a to a′, and the image point b from b to b′. The image will be free from distortion when the distances Δ_1 and Δ_2 are of equal magnitude. However, from

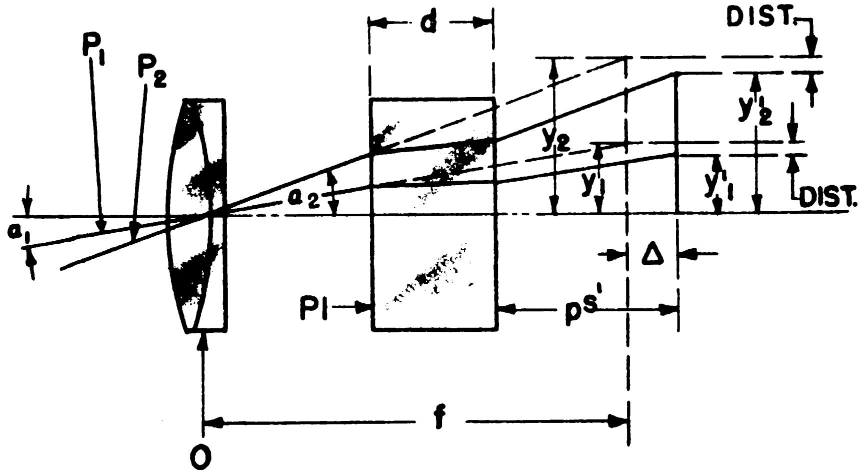


Figure 41. Distortion caused by a plane-parallel plate.

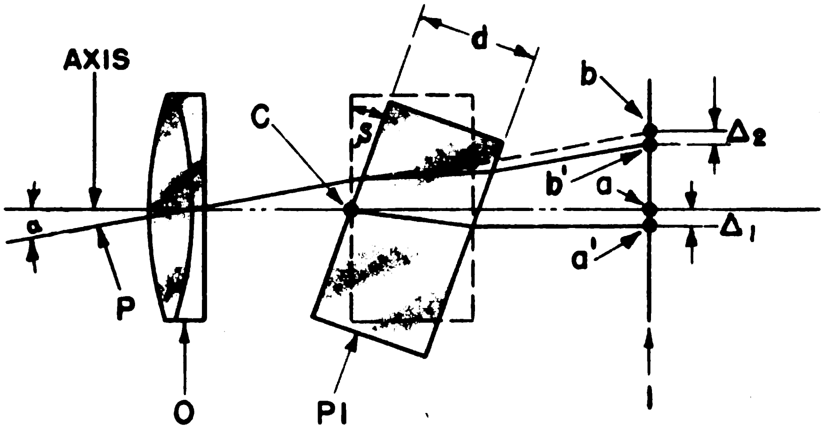


Figure 42. The tilting plane-parallel plate.

figure 42 it may be seen that Δ_2 is greater than Δ_1 and, hence, the image is distorted The actual amount of distortion is equal to the difference in the values of Δ_2 and Δ_1. These values are computed with the following equations:

$$\Delta_1 = d\ (\sin\ \delta) \left\{ \frac{\cos\ \delta}{\sqrt{n^2 - \sin^2 \delta}} - 1 \right\} \qquad (60)$$

$$\Delta_2 = d\ [\sin\ (\delta + \alpha)] \left\{ \frac{\cos\ (\delta - \alpha)}{\sqrt{n^2 - \sin^2 (\delta - \alpha)}} - 1 \right\} \qquad (61)$$

$$\text{DIST} = \Delta_2 - \Delta_1 \qquad (62)$$

Numerical Example: A plane-parallel plate, 0.25 inch thick, is placed between the objective and the image plane. The principal ray and the axis include an angle of 2°. What is the magnitude of the distortion when the plate is tilted through an angle of 1°?

We have: $d = 0.25$, $n = 1.5165$, $\delta = 1°$, and $\alpha = -2°$.

Substitute these values into equations (60) to (62) we find:

$$\Delta_1 = -0.001486$$
$$\Delta_2 = -0.004464$$
$$\text{DIST} = -0.002978$$

32. Test for Planeness

The planeness of a plane-parallel plate is tested with the aid of a true plane test plate. The surfaces of the plate to be tested and the test plate are cleaned and carefully freed from dust with a camel's hair brush. The cleaned surfaces are then brought into contact so that only a very thin wedge-shaped film of air is formed between them (fig. 43(a)). When viewed in ordinary light there will be seen a series of colored bands across the plate. These bands are known as the *"Newton interference bands."* If the surface under test is irregular, the pattern assumes the appearance of a contour map If the colored bands appear straight, parallel, and equally spaced as shown in figure 43(b), the surface is plane. On the other hand, if the colored bands appear as perfect circles (see fig. 43(c)), the surface of the plate is slightly spherical. When, upon a slight pressure on the plate, the concentric bands move away from the center, the surface is convex and vice versa.

If the monochromatic light source L is used during the test, a diffusing screen S is placed between the light source and the plates as illustrated in figure 43(a). In this case there will be seen a series of light and dark bands across the surface. At the first dark band the layer of air between the two plates is equal to one-half of a wave-length of the light employed in the test, one whole wave-length at the second band, etc. Hence, if helium light is used whose wave-length $\lambda = 0.00002309$ inches, the thicknesses of the air layers between the plates are equal to:

0.00001155 inches at the 1st dark band,
0.00002309 inches at the 2nd dark band,
0.00003464 inches at the 3rd dark band,
0.00004618 inches at the 4th dark band, etc.

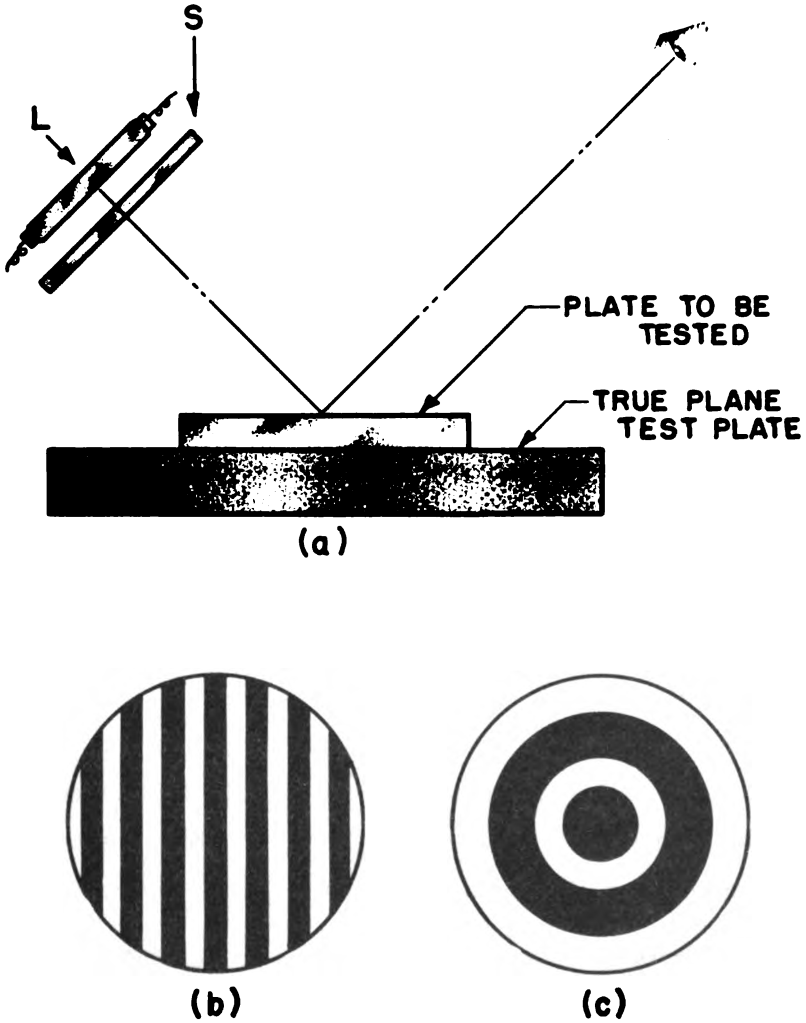


Figure 43. Testing a plane-parallel plate for planeness.

33. Test for Parallelism

Thin plane-parallel plates, such as reticles, are examined for parallelism with an instrument known as the *"interferometer."* In figure 44(a) is illustrated the optical system of the Peters–Page interferometer. The helium tube T is mounted in the rear focal plane of the objective L_0 and immediately in front of the tube is the diaphragm D_1. Light from the tube T will emerge from the objective L_0 in parallel bundles and, after a total reflection in the prism P, it will proceed toward the reticle R, which is placed on the table in the front focal plane of the objective. The surfaces of the reticle will reflect the light rays back and the objective L_0 and the collective lens L_1 will bring the rays to a focus in the front focal plane of the eyepiece L_2. The eye, placed behind the diaphragm D_2, can conveniently view the interference patterns. If the two surfaces of the reticle are parallel to each other there will be seen just one light band extending across the entire surface. If a number of light and dark bands are visible the two surfaces are not parallel to each other but will include a small angle designated by the letter θ in figure 44(b). The magnitude of this angle, that is, the deviation from parallelism, is determined with the following formula:

$$\tan \theta = \frac{N\lambda}{2nD} \tag{63}$$

and the difference between the two extreme thicknesses of the reticle is:

$$\Delta d = d_2 - d_1 = D \tan \theta \tag{64}$$

The following abbreviations are used in these equations:

D = the distance between the centers of the first and the last dark band in the interence pattern.

d_1 = the thickness of the reticle at the center of the first dark band.

d_2 = the thickness of the reticle at the center of the last dark band.

N = the number of the dark bands seen in the interference pattern.

n = the refractive index of the glass of which the reticle is made.

Example: The slightly wedge-shaped reticle shown in figure 44(b) is made of a baryta light flint glass. When examined with the interferometer under helium light, there are seen six dark bands in the interference pattern. The distance between the first and the sixth dark band is 25 millimeters. What is the deviation from parallelism? We have, $n = 1.5838$, $N = 6$, $D = 25$ mm $= 0.984$ inch, and $\lambda = 0.0005876$ mm $= 0.00002309$ inch. Substituting these values into equations (63) and (64) we find:

$\tan \theta = 0.0000444$; hence, $\theta = 9$ seconds of arc and, $\Delta d = 0.0000437$ inches.

The interference patterns also indicate whether or not the surfaces of the reticle are plane. They are plane if the bands appear straight, parallel, and equally spaced.

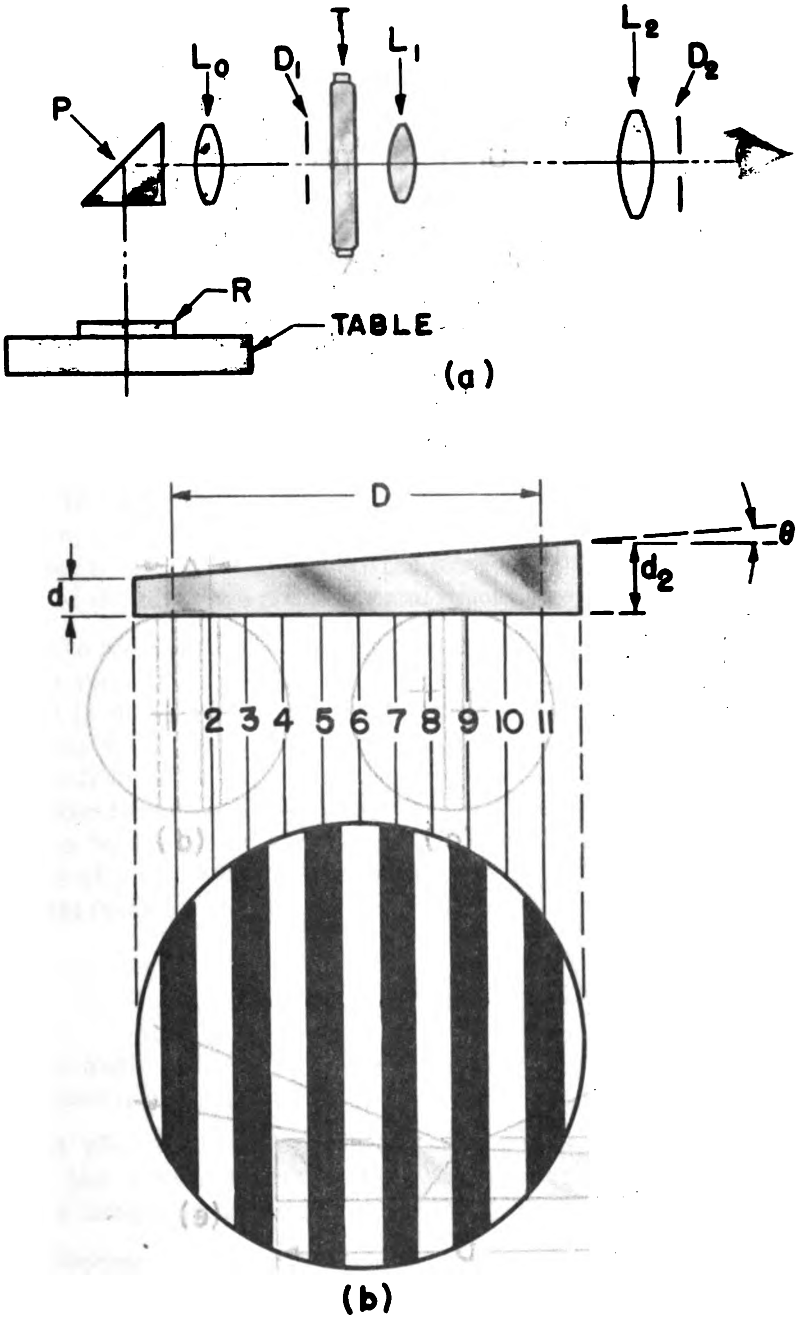


Figure 44. The Peters-Page interferometer.

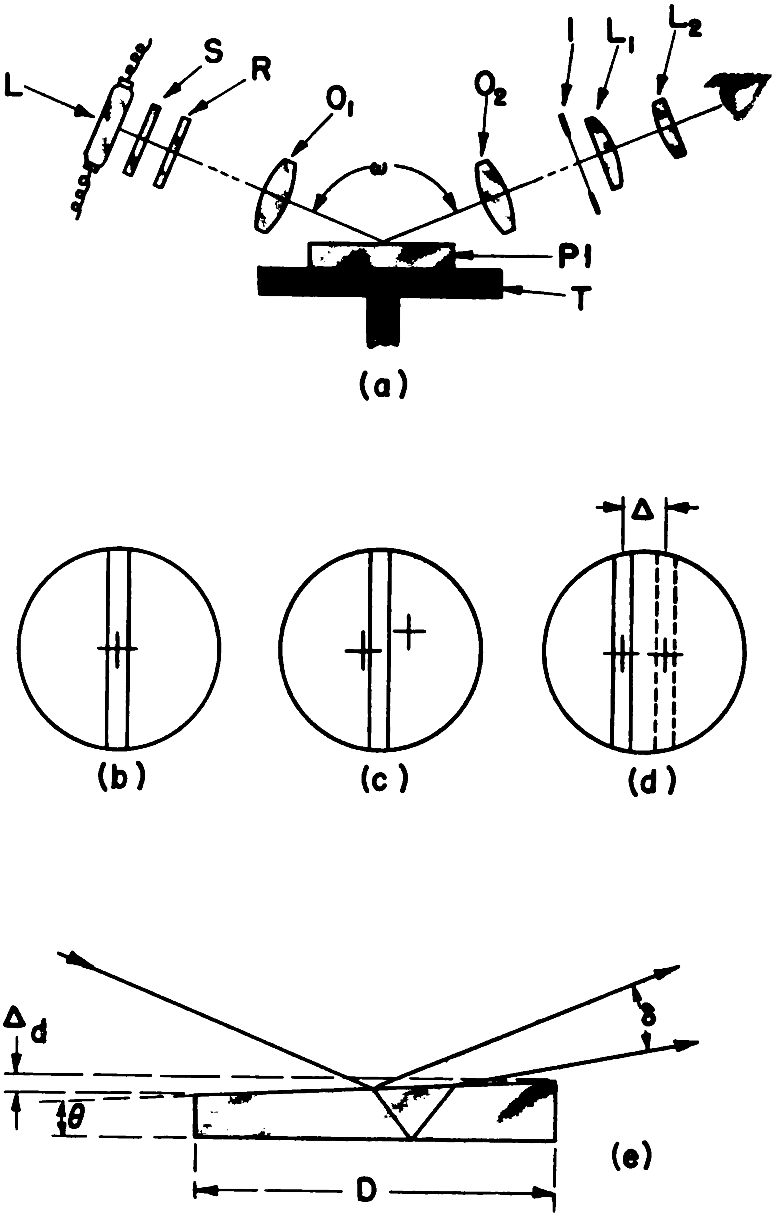


Figure 45. Method of testing plane-parallel plates for planeness and parallelism.

Sodium light may be used if the thickness of the plate does not exceed 2 mm. The green light ($\lambda = 0.000546$ mm) and the dark blue light ($\lambda = 0.000436$ mm) of the mercury tube can be used for thicknesses up to 10 mm.

The parallelism of thick plates is tested with an instrument whose essential optical components are shown in figure 45. It consists of a collimator and an observing telescope placed in such a way that the angle ω between them is about 135°. The collimator contains the objective O_1, the graduated reticle R, and the light-source L. Between the light and the reticle R is the diffusing screen S. The observing telescope contains the objective O_2, two parallel vertical wires at I, and the micrometer eyepiece consisting of the lenses L_1 and L_2. The eyepiece is equipped with a micrometer drum on which can be read the lateral displacement Δ of the vertical wires. The plate P1 to be tested rests on the turntable T. On the reticle of the collimator there is usually etched a cross. The telescope and the collimator are adjusted so that the image of the cross appears sharp and distinct between parallel vertical wires of the collimator as illustrated in figure 45(b). In order that image appears on the axis of the telescope, it may become necessary to either increase or decrease the angle ω. When upon looking into the telescope only one well defined image of the cross is visible, then the two surfaces of the plate P1 are not only parallel to each other, but they are also flat or plane. If two clear images of the cross are seen, then the surfaces of the plate are plane, but the plane is slightly wedge-shaped as shown in figure 45(c). If the image of the cross appears broad and blurred, the surfaces of the plate are not plane and must be repolished.

In order to measure the deviation from parallelism, turn the table T (on which rests the plate P1) until the horizontal bars of the two images of the cross appear in one plane as illustrated in figure 45(d). Turn the micrometer drum until one image of the cross is visible between the two vertical wires in the eyepiece and read the scale on the drum. Now turn the drum until the second image of the cross appears between the vertical wires and take a second reading on the drum. The difference in the two readings is equal to the distance Δ between the two images. The focal length f of the objective O_2 is known; hence, we are able to determine the magnitude of the angle θ, that is, the angle included by the two surfaces of the plate P1 as follows:

$$\tan\theta = \frac{\tan\delta}{2n} = \frac{\Delta}{2nf} \qquad (65)$$

where δ is the angle included by the rays reflected on the front and on the rear surfaces of the plate as shown in figure 45(e).

Example: A plate made of borosilicate crown glass has a diameter of 3 inches. The objective O_2 has a focal length of 10 inches. What is the deviation from parallelism when the two images of the cross are separated by a distance of 0.00147 inches? We have:

$D = 3$ inches, $f = 10$ inches $n = 1.5170$, and $\Delta = 0.00147$ inches

Substitute these values into equations (65) and (64) and find,

$\tan\theta = 0.00004845$ $\theta = 10$ seconds of arc $\Delta d = 0.000145$ inches

SECTION IV

PRISMS

34. Introduction

In this section are described the optical characteristics of prisms and some of their uses in fire-control instruments. Many formulas useful in the design of optical instruments are given. Only rays lying in a *"PRINCIPAL SECTION"* of a prism, which is a section formed by any plane perpendicular to the edge of the system, are considered.

In section IX of this volume are given diagrams of prisms or prism systems frequently used in fire-control instruments. Dimensions are given in terms of the diameter of a cylindrical beam of light which can be transmitted by the prism or prism system.

35. Symbols and Sign Conventions

The refractive indices of the media are designated as follows:

n_o = for medium in front of the first prism face.

n_1 = for medium between the two prism faces.

n_2 = for medium behind the second prism face.

J = the critical angle, or the angle of total reflection between two media.

θ = the refracting angle of the prism. It is considered positive when the second face of the prism is turned clockwise in order to meet the first prism face, and vice versa.

δ_v = the angle of deviation due to refraction on the vth surface of the prism. It is equal to the difference between the angle of incidence and the angle of refraction at the vth surface.

Δ = the angle of total deviation caused by a prism. This angle is formed by the incident ray continued forward and the emerging ray. It is reckoned positive when the continuation of the incident ray is moved clockwise in order to meet the emerging ray, and vice versa.

i_v = the angle of incidence at the vth surface of the prism. This angle is considered positive when the incident ray is moved clockwise toward the normal to the vth surface, and vice versa.

i_v' = the angle of refraction at the vth surface. It is reckoned positive when the refracted ray is turned clockwise toward the normal to the vth surface, and vice versa.

36. General Formulae

Reference is made to figure 46 for deviation of the following formulae:

$$\sin i_1' = \frac{n_o \sin i_1}{n_1} \qquad (66)$$

$$i_2 = i_1' - \theta \qquad (67)$$

$$\sin i_2' = \frac{n_1 \sin i_2}{n_2} \qquad (68)$$

$$\delta_1 = i_1 - i_1' \tag{69}$$

$$\delta_2 = i_2 - i_2' \tag{70}$$

$$\Delta = \delta_1 + \delta_2 = i_1 - i_2' - \theta \tag{71}$$

$$\sin i_2' = \cos \theta \sin i_1 - \sin \theta \sqrt{n_1^2 - \sin^2 i_1} \tag{72}$$

If the refracting angle θ, the angle of emergence i_2' and the refractive indices n_v are known, we can compute the magnitude of the angle of incidence by an equation similar to equation (72) which is effected by interchanging i_2' and i_1 in equation (72). This equation is:

$$\sin i_1 = \cos \theta \sin i_2' + \sin \theta \sqrt{n_1^2 - \sin^2 i_2'} \tag{73}$$

In figure 46 are illustrated the paths of light through three refracting prisms. In diagram (a) and (b) the prisms are surrounded by air, and the one shown in diagram (c) by water. The prisms shown in figure 47 are surrounded by air and have the same refracting angle. In diagram (a) the entrance surface is normal to the axis, and in diagram (b) the exit surface is normal to the axis. These five prisms have the following constants in common:

$n_1 = 1.5170$, $\theta = 30°$, $\theta = -30°$ for prism shown in figure 46(b).

$|i_1| = 15°$ from geometry of figures.

$n_0 = n_2 = 1.3333$ for prism shown in figure 46(c).

$n_0 = n_2 = 1$ for prisms shown in figures 46(a) and (b).

Substituting these values into equations (66) to (71) we obtain the results tabulated here:

	Figures				
	46(a)	46(b)	46(c)	47(a)	47(b)
$i_1 =$	+ 15°	— 15°	+ 15°	+ 15°	+ 15°
$i_1' =$	+ 9°49′24.2″	— 9°49′24.2″	+13°8′55.0″	+ 9°49′24.2″	+ 9°49′24.2″
$i_2 =$	— 20°10′35.8″	+ 20°10′35.8″	— 16°51′5.0″	— 20°10′35.8″	— 20°10′35.8″
$i_2' =$	— 31°32′58.6″	+ 31°32′58.6″	— 19°15′30.6″	— 31°32′58.6″	— 31°32′58.6″
$\Delta =$	+ 16°32′58.6″	— 16°32′58.6″	+4°15′30.6″	+ 16°32′58.6″	+ 16°32′58.6″

It may be seen that the absolute value of Δ for the prisms in air remains constant no matter how the prism is placed providing the magnitude of the angle i_1 remains constant.

37. Case of Minimum Deviation

$$\sin i_1 = -\sin i_2' = \frac{n_1}{n_0} \sin \frac{\theta}{2} \tag{74}$$

$$\sin i_1' = -\sin i_2 = \sin \frac{\theta}{2} \tag{75}$$

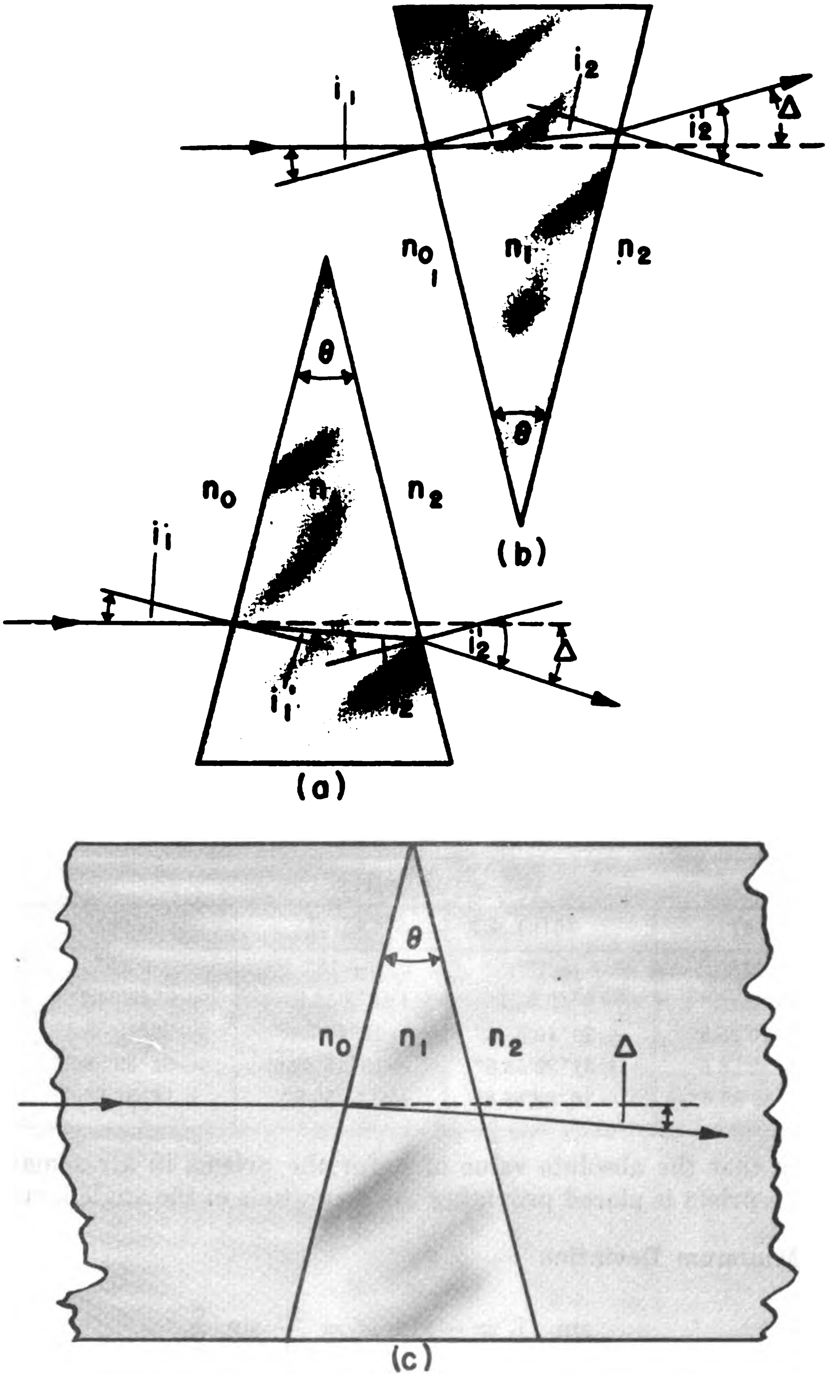


Figure 46. General cases.

$$\delta_1 = \delta_2 = i_1 - i_1' \tag{76}$$

$$\Delta = 2i_1 - \theta \tag{77}$$

$$\frac{\tan i_1}{\tan i_1'} = \frac{\tan i_2'}{\tan i_2} \tag{78}$$

$$\sin \frac{\theta + \Delta}{2} = \frac{n_1}{n_o} \sin \frac{\theta}{2} \tag{79}$$

If the prism is surrounded by air as the one shown in figure 48(a) then, $n_o = 1$, and equation (79) may be written as follows:

$$\sin \frac{\theta + \Delta}{2} = n_1 \sin \frac{\theta}{2} \tag{80}$$

From this equation we derive,

$$n_1 = \frac{\sin \frac{\theta + \Delta}{2}}{\sin \frac{\theta}{2}} \tag{81}$$

Fraunhofer made this equation the basis for his method for the determination of the refractive index of a transparent medium.

The case of minimum deviation is illustrated in figure 48, where the prism in diagram (a) is surrounded by air ($n_o = n_2 = 1$), while the prism in diagram (b) is surrounded by water ($n_o = n_2 = 1.3333$) $n_1 = 1.5170$ and $\theta = 30°$ in both cases. Substituting these values into the above equations we obtain the results tabulated here:

	Fig. 48(a)	Fig. 48(b)
$i_1 = -i_2' =$	+ 23°7′5.2″	+ 17°7′34.7″
$i_1' = -i_2 =$	+ 15°	+15°
$\Delta =$	+ 16°14′10.3″	+4°15′9.4″

38. Case of Normal Incidence

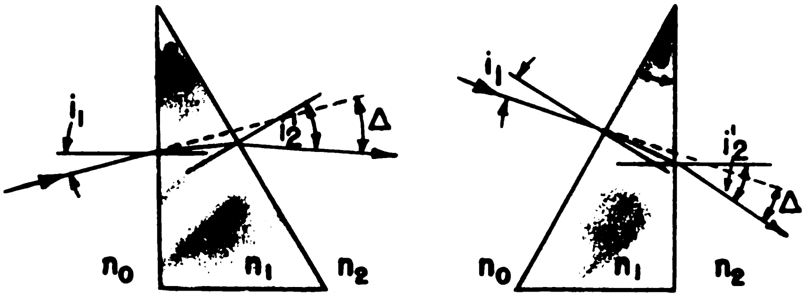
In the case of normal incidence (fig. 49) we make use of the equations following:

$$i_1 = i_1' = 0 \tag{82}$$

$$i_2 = -\theta \tag{83}$$

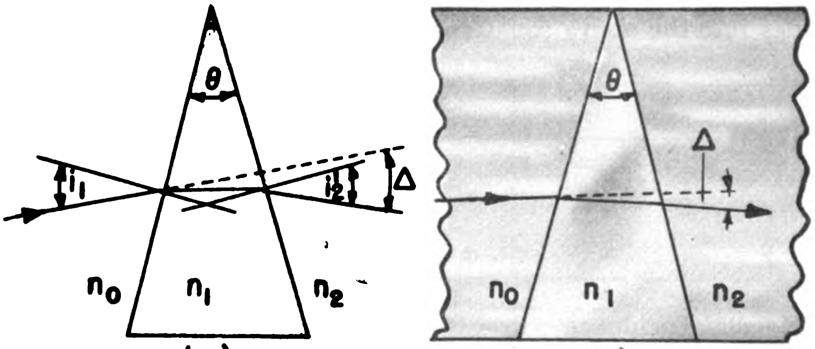
$$\sin i_2' = \frac{n_1 \sin i_2}{n_2} \tag{84}$$

$$\delta_1 = 0 \tag{85}$$



(a) (b)

Figure 47. General cases.



(a) (b)

Figure 48. Case of minimum deviation.

$$\delta_2 = i_2 - i_2' \tag{86}$$

$$\Delta = -\theta - i_2' \tag{87}$$

$$n_1 = \frac{n_2 \sin (\theta + \Delta)}{\sin \theta} \tag{88}$$

The prisms shown in figure 49 have the following constants in common:

$$n_o = n_2 = 1 \qquad n_1 = 1.5170 \qquad \theta = 30°$$

Substituting these values into equations (82) to (88) we find:

$$i_1 = i_1 = 0 \qquad i_2 = -30° \qquad i_2' = -49°19'55.6''$$

$$\Delta = +19°19'55.6''$$

39. Case of Normal Emergence

If the ray shall emerge from the prism normally to the second surface (fig. 50) we determine the constants with the following equations:

$$\sin i_1 = \frac{n_1 \sin \theta}{n_o} \tag{89}$$

$$i_1' = \theta \tag{90}$$

$$i_2 = i_2' = 0 \tag{91}$$

$$\delta_1 = i_1 - i_1' \tag{92}$$

$$\delta_2 = 0 \tag{93}$$

$$\Delta = i_1 - \theta \tag{94}$$

$$n_1 = \frac{n_o \sin (\theta + \Delta)}{\sin \theta} \tag{95}$$

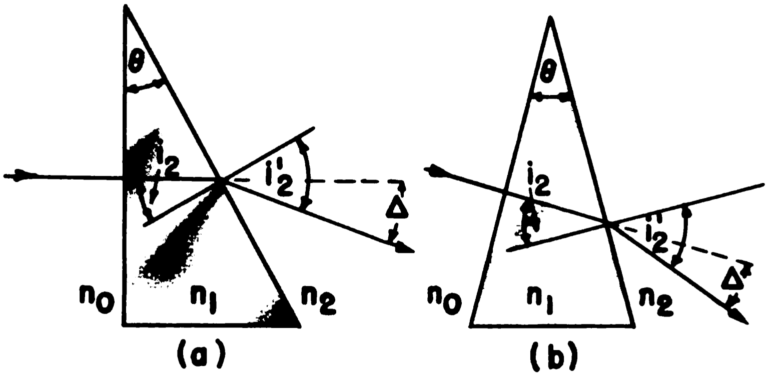
The case of normal emergence is illustrated in figure 50 where both prisms are assumed to be in air. The refracting angle of each prism is equal to 30° and the refractive index is 1.5170. Substituting these values into equations (89) to (95) we find,

$$i_1 = +49°19'55.6'' \qquad i_2 = i_2' = 0$$

$$i_1' = +30° \qquad \Delta = +19°19'55.6'' \qquad n_1 = 1.5170$$

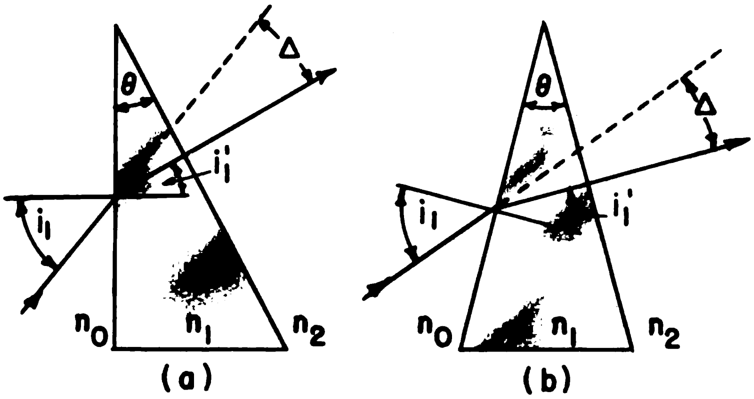
40. Dispersion by Prisms

In the preceding paragraphs it has been tacitly assumed that the index of refraction of the glass is constant. This assumption is permissible so long as we deal only with light of a definite color or wavelength; in the preceding examples we have considered



(a) (b)

Figure 49. Case of normal incidence.



(a) (b)

Figure 50. Case of normal emergence.

yellow light, that is, light corresponding to the D line of the spectrum. However, the index of refraction has a different value for each color, and the deviation caused by a prism will vary for each color as illustrated in figure 51(a). When a ray of white light, proceeding from the luminous point P enters the prism, each color present in the ray is refracted through a different angle, that is, the light is *"DISPERSED."* After another refraction at the second surface the rays will emerge from the prism and then they will spread out fanwise. The red rays are deviated the least, the violet the most, and between these rays there will travel the rays of the remaining colors of the spectrum. The observer will see an image of the luminous point P in the direction of the point P′. The image will appear as an elongated streak, showing all colors of the spectrum as indicated in figure 51(b). The angle Γ between the emerging red and violet rays is called the angular chromatic aberration of the prism. If the prism is to be achromatic then the following condition must be satisfied:

$$\Gamma = {}_{C}\Delta - {}_{F}\Delta = 0 \tag{96}$$

that is, the red and blue rays will emerge from the prism traveling parallel to each other. This condition can never be realized with a single prism but requires two prisms. For example, one made of crown glass and another of flint glass may be cemented together as shown in figure 52. Let us compute an achromatic prism. The glasses used will have the following characteristics:

Prism	Glass	Δn	n_C	n_D	n_F	ν
1	Crown	0.00806	1.51369	1.5161	1.52175	64.0
2	Flint	0.01713	1.61622	1.6211	1.63335	36.3

First, we determine the angular deviation for the red and blue rays caused by the single crown-glass prism. The refracting angle is $\theta_1 = 30°$; hence, in the case of minimum deviation we have $i_1 = 23°6'13.0''$. Apply equations (74) to (77) as follows:

$\lambda =$	C	D	F
$\sin i_1' =$	0.259231	0.258819	0.257858
$i_2 =$	$-14°58'32.2''$	$-15°0'0''$	$-15°3'25.2''$
$i_2' =$	$-23°1'33.6''$	$-23°6'13.0''$	$-23°17'9.2''$
$\Delta_1 =$	$+16°7'46.6''$	$+16°12'25.7''$	$+16°23'22.2''$

and $\Gamma_1 = {}_{C}\Delta_1 - {}_{F}\Delta_1 = -0°15'35.8''$

We combine this prism with the flint glass prism and determine the new value of i_2' with equation (68), and then we determine the value of θ_2 with the following equation:

$$\cot\text{an}\ \theta_2 = -\left\{ \frac{\Delta n_2}{2\Delta n_1 \ (\sin \theta/2) \cos i_2'} + \tan i_2' \right\} \tag{97}$$

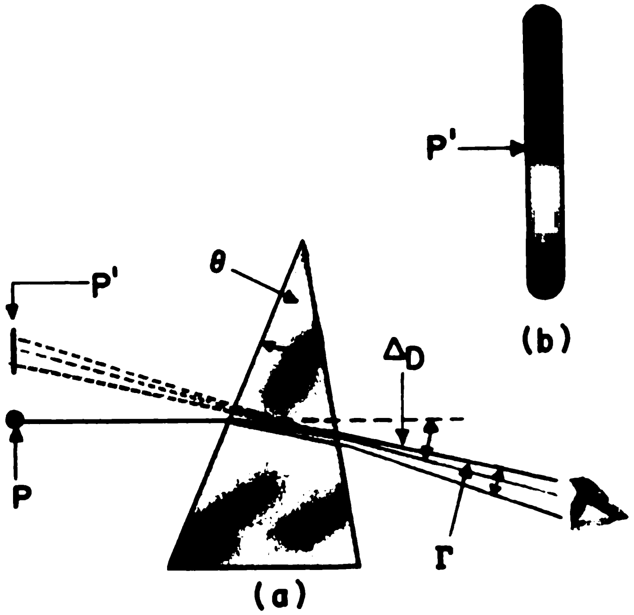


Figure 51. Dispersion caused by a prism.

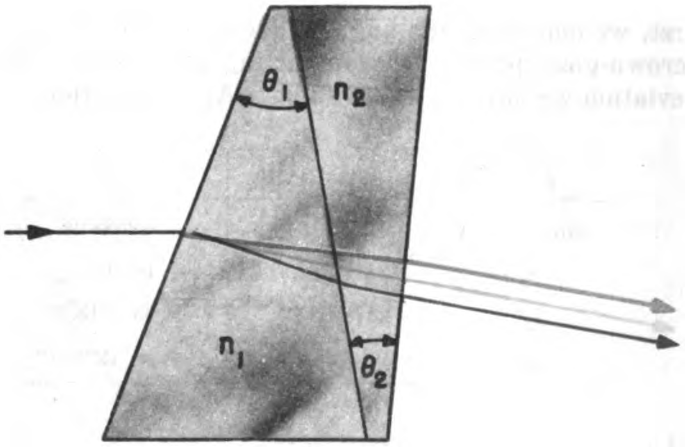


Figure 52. The achromatic prism.

In this equation the value of i_2' refers to the D-line of the spectrum. Then, applying the following equations we obtain the tabulated results.

$$\left.\begin{cases} i_3 = i_2' - \theta_2 \\ i_3' = \dfrac{n_2 \sin i_3}{n_3} \\ \Delta = i_1 - i_3' - (\theta_1 + \theta_2) \\ \Gamma = {}_{C}\Delta - {}_{F}\Delta \end{cases}\right\} \tag{98}$$

$\lambda =$	C	D	F
$i_2' =$	$-14°0'19.8''$	$-14°0'28.4''$	$-14°0'23.0''$
$i_3 =$	$+0°5'28.3''$	$+0°5'19.7''$	$+0°5'25.1''$
$i_3' =$	$+0°8'50.6''$	$+0°8'38.0''$	$+0°8'51.0''$
$\Delta =$	$+7°3'10.4''$	$+7°3'23.0''$	$+7°3'10.1''$

$\Gamma = +0°0'0.4''$

There is practically no primary chromatic aberration and the prism system may be considered achromatic. However, the angular differences between the red and yellow rays, and the blue and yellow rays in the secondary chromatic aberration amounts to 12.9 seconds of arc.

41. Wedge Prisms

In various types of range finders are employed prisms with small refracting angles. These prisms, known as *"wedge prisms,"* are used for measuring the range, and also for the purpose of adjusting the halving mechanism. Such a wedge is shown in figure 53 (a). Since the angles are small we can assume that the angle and their sines are identical, and the formulae for computing the magnitudes of the refracting angle and the angle of deviation may be written as follows:

$$\Delta = (n - 1)\,\theta, \quad \text{hence,} \tag{99}$$

$$\theta = \frac{\Delta}{n - 1} \tag{100}$$

A prism thus computed is a prism of minimum deviation. For example, the refracting angle of a wedge prism is equal to 1°56′45″, and the refractive index of the glass is 1.5165. What is the deviation caused by this wedge?

We have, $n = 1.5165$ and $\theta = 1°56'45'' = 116.75'$.

Substituting these values into equation (98) we find:

$$\Delta = 0.5165 \times 116.75 = 60.3014' = 1°0'18.1''$$

As a rule, the wedges will be mounted in the instrument with the front surface normal to the axis of the instrument, that is, we have the case of normal incidence. In this case we determine the deviation of the prism with equations (82) to (87) and find:

$$\Delta = 1°0'20.9''$$

If n and Δ are given we determine the magnitude of the refracting angle from equation (88) as follows:

$$\sin \theta = \frac{\sin \Delta}{\sqrt{n^2 + 1 - 2n \cos \Delta}} \tag{101}$$

In long base range finders the wedge prisms are made achromatic as illustrated in figure 53(b). In this case the following three conditions must be satisfied:

(a) $i_1 = 0$, that is, the light must enter the prism normally to the first refracting surface.

(b) ${}_C\Delta = {}_F\Delta$, that is, the prism must be achromatic with respect to the red and blue light.

(c) ${}_D\Delta =$ constant, that is, the prism must have a definite deviation for the yellow light.

These conditions will be fulfilled as soon as the following two equations are satisfied:

$$\Delta n_1 \sin \theta_1 \left\{ (\sin i_2' + \cos i_2' \operatorname{cotan} \theta_2) \right\} + \Delta n_2 = 0 \tag{102}$$

$$n_2 \cos i_2' \tan \theta_2 - \cos (\theta_1 + {}_D\Delta) \tan \theta_2 - n_2 \sin i_2' - \sin (\theta_1 + {}_D\Delta) = 0 \tag{103}$$

We determine the value of θ_2 from equation (102) and then substitute this value into equation (103) and we obtain an equation of the 6th order for $\sin \theta_1$.

For example, an achromatic wedge prism is to be made of the same glasses as the achromatic prism computed in the preceding paragraph. The required deviation for the yellow light is ${}_D\Delta = 1°45'$. Substituting this value into the equations (102) and (103) we find:

$$\theta_1 = +\,7°44'44'' \text{ and } \theta_2 = -\,3°37'43''$$

42. Reflections and Refractions in Right-Angle Prisms

In figure 54 are shown the paths through a right-angle reflecting prism of three rays designated by the letters A, B, and C, respectively. The axial ray A enters the prism normally to the front surface, and will then proceed to the reflecting surface. Here it will be totally reflected toward the rear surface where it will emerge from the prism normally to the exit surface. Therefore, the ray has been deviated through an angle of 90°. We have,

$$\theta_1 = \theta_2 = 45° \quad \text{and} \quad R = 90°$$

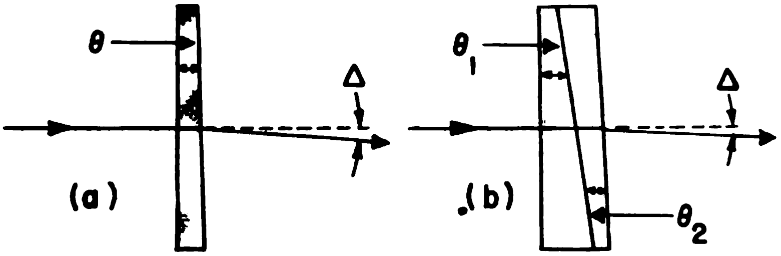


Figure 53. The wedge prism.

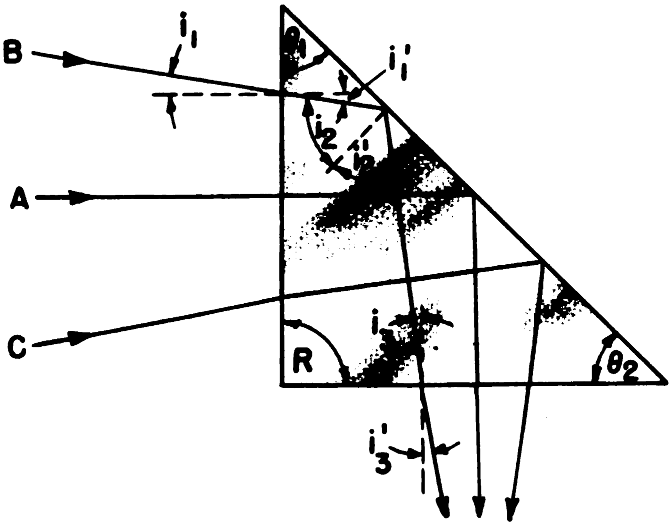


Figure 54. Light-path through a right-angle prism.

Hence, in the case of the ray A we have:

$$i_1 = i_1' = i_3 = i_3' = 0 \qquad i_2 = -i_3 = \theta_1 = \theta_2 = 45°$$

$$\Delta = \theta_1 + \theta_2 = 90°$$

For the rays B and C the angle of incidence is equal to ± 10° and the refractive index $n = 1.5170$. Then, in the case of ray B we find:

$$i_1 = -10° \qquad \sin i_1' = \frac{\sin i_1}{n} \qquad i_1' = -6°34'22.6''$$

$$i_2 = -i_2' = i_1' - \theta_1 = -51°34'22.6''$$

$$i_3 = -\theta_2 - i_2' = +6°34'22.6''$$

$$i_3' = n \sin i_3 = +10° \qquad \Delta = \theta_1 + \theta_2 = 90°$$

In the case of ray C we have:

$$i_1 = +10° \qquad i_2 = -i_2' = i_1' - \theta_1 = -38°25'37.4''$$

$$i_1' = +6°34'22.6'' \qquad i_3 = -6°34'22.6''$$

$$i_3' = -10° \qquad \Delta = 90°$$

Applying equation (7) we find that the critical angle for this glass is equal to 41°14′19″. It will be noted that in the case of ray C the angle of incidence on the reflecting surface is $i_2 = -38°25'37.4''$, that is, the angle is smaller than the critical angle J and, therefore, we must either silver the reflecting surface or replace the glass by one which has a refractive index of not less than $n = 1.61$ (see table 2 in section I).

The importance of the critical angle J in telescope design will be illustrated by a few practical examples.

In figure 55 is shown the right-angle prism P placed between the objective O and the reticle of a telescope. The two rays inclined toward the axis at an angle α, and intersecting the optical axis at the first principal plane of the objective, are termed the "*FIELD RAYS.*" The aperture of the reticle must be large enough to permit the passage of these rays. Hence, 2α is the range of observation, or the "*TRUE FIELD OF VIEW*" of the telescope. It is evident that if we increase the true field of the telescope, we will reach a point where the angle of incidence i_2 at the reflecting surface of the prism will be equal to, or even smaller than the critical angle J. In these cases a portion of the light will emerge from the prism and enter into air. Consequently, the outer field of the image will appear much fainter than the central portion. This can be avoided by silvering the reflecting surface of the prism; however, silvered surfaces are not desirable in fire control instruments.

The maximum semi-field of view we can give a telescope employing right-angle prism, which is to reflect all the rays without silvering the reflecting surface of the prism, is computed with the following equation:

$$\sin \alpha = -n \sin (J - \theta) \tag{104}$$

Substituting equation (7a) into this equation we have:

$$\sin \alpha = -\cos \theta + \sin \theta \sqrt{n^2 - 1} \qquad (105)$$

In a right-angle prism $\theta = 45°$ and, if the true field of view 2α of the telescope is given, we may determine the value n for which total reflection will occur with the aid of the following equation:

$$n = \frac{\sqrt{\sin \alpha (\sin \alpha + 2 \cos \theta) + 1}}{\sin \theta} \qquad (106)$$

$$n = \frac{\sqrt{\sin^2 \alpha + 1.4142136 \sin \alpha + 1}}{0.7071068}$$

Example:

A telescope, employing a Porro prism erecting system, is to have a true field of view of 16°30′. What is the lowest refractive index the prism glass can have in order to obtain total reflection with unsilvered reflecting surfaces?

$$\alpha = 8°15' \quad \text{and} \quad \theta = 45°$$

hence, from equation (106)

$$n = 1.5643$$

Let us determine the maximum value of 2α a telescope can have when right-angle prisms with unsilvered surfaces are employed. For n we substitute the values varying between the limits of 1.5 and 1.7 into equation (105). The results of these computations are given in table X.

TABLE X

MAXIMUM FIELD OF VIEW OF A PRISM TELESCOPE
(UNSILVERED REFLECTING SURFACES)

n	2α	n	2α	n	2α
1.50	9°34′31″	1.57	17° 6′22″	1.64	24°28′55″
1.51	10°39′49″	1.58	18°10′ 3″	1.65	25°31′38″
1.52	11°44′51″	1.59	19°13′33″	1.66	26°34′15″
1.53	12°49′37″	1.60	20°16′54″	1.67	27°36′48″
1.54	13°54′ 9″	1.61	21°20′ 6″	1.68	28°39′17″
1.55	14°58′25″	1.62	22°23′10″	1.69	29°41′42″
1.56	16° 2′30″	1.63	23°26′ 5″	1.70	30°44′ 5″

In the case of wide-angle binocular field glasses the first Porro prism P (see fig. 56) is separated from the objective O by a relatively short distance. The two field rays, including the true field of view 2α, are the extreme field rays which determine the size of the image and, therefore, the aperture of the reticle. The two rays, shown in blue, traveling parallel to the optical axis before entering the objective are called *"rim rays."*

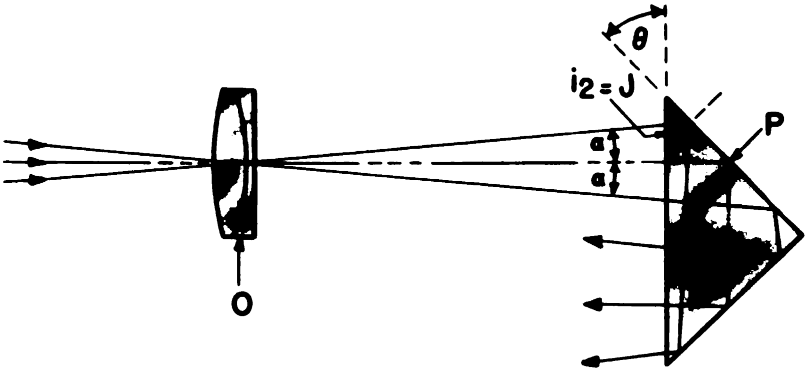


Figure 55. Total reflections in a right-angle prism.

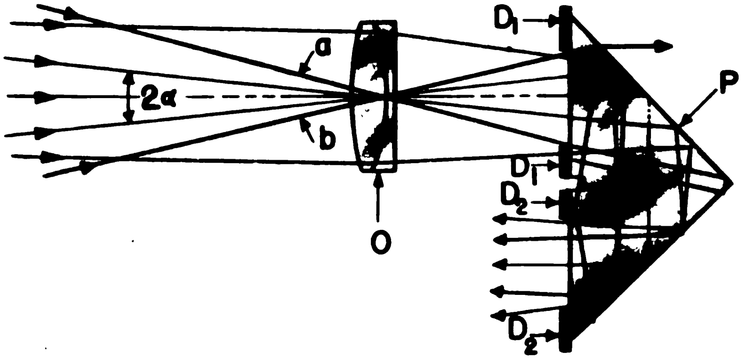


Figure 56. Case of three reflections in a right-angle prism.

These rim rays are separated by a distance EP which is equal to the diameter of the entrance pupil of the telescope. The rim rays determine the diameter AP of the exit pupil of the instrument and, therefore, the light transmission of the telescope. The prisms must be made large enough to permit the passage of the extreme field rays and the rim rays. As a rule, a diaphragm is placed at the entrance surface of the prism. This is designated by letters $D_1 - D_1$ in figure 56. It will be noted that the rim rays enter the prism at a much greater distance from the axis than the extreme field rays; hence, we are unable to prevent field rays coming from objects outside the true field of view from entering the prism. Two such rays, shown in red, traveling outside the true field, are designated by the letters a and b, respectively. Let us trace the paths of these rays through the prism. The ray a will enter the prism and, after a reflection on each reflecting surface of the prism, it will emerge from the prism at the entrance surface. Here the diaphragm $D_1 - D_1$ will stop its further progress towards the image plane of the objective. The ray b will enter the prism and proceed toward the first reflecting surface. The angle of incidence on this surface is smaller than the critical angle J and, therefore, the major portion of the light becomes harmless by emerging from the prism into air. The other portion of the light will be reflected back towards the entrance surface, and, after a reflection on this surface and another at the second reflecting surface, it will emerge at the entrance surface. The diaphragm $D_2 - D_2$ will prevent this ray from entering the second Porro prism (not shown in the diagram). If these rays were permitted to reach the image plane they would produce a *"reflection image,"* better known to the man in the shop as *"prism ghosts."* These images are, as a rule, inverted. If these prism ghosts are large images of very bright targets outside the true field of view, then the image of the real target can hardly be seen. The diaphragms $D_1 - D_1$ and $D_2 - D_2$ of proper apertures will eliminate a number of these images. Cutting a groove or slit S across the entrance surface of the prism will, in most cases, eliminate all the prism ghosts.

Similar conditions exist when a right-angle prism receives the light first as, for example, the objective prism in the Battery Commander's Telescope. In this case nearly all the reflection images can be eliminated by placing a diaphragm behind the exit surface of the prism.

43. Rotating Prism System

Rotating prism systems are used in instruments employed by the observer to view the entire horizon without changing his position. These systems are used in all panoramic sights, and also in periscopes for submarines.

All rotating prisms consist of two prisms, a right-angle prism followed by a:

Harting-Dove Prism (see fig. 57), or
Twin Prism (see fig. 58), or
Pechan Prism (see fig. 59)

In the Goerz and Zeiss systems both prisms are placed in front of the objective, since the Harting-Dove prism as well as the Twin prism must be located so that they receive only parallel beams of light. The Pechan prism may be placed either in front or behind the objective. The reflecting surface of each rotating prism acts like a plane rotating mirror, that is, when the prism is turned through an angle α the image will be turned through an angle of 2α. Therefore, in order to prevent the image from *"falling"* when

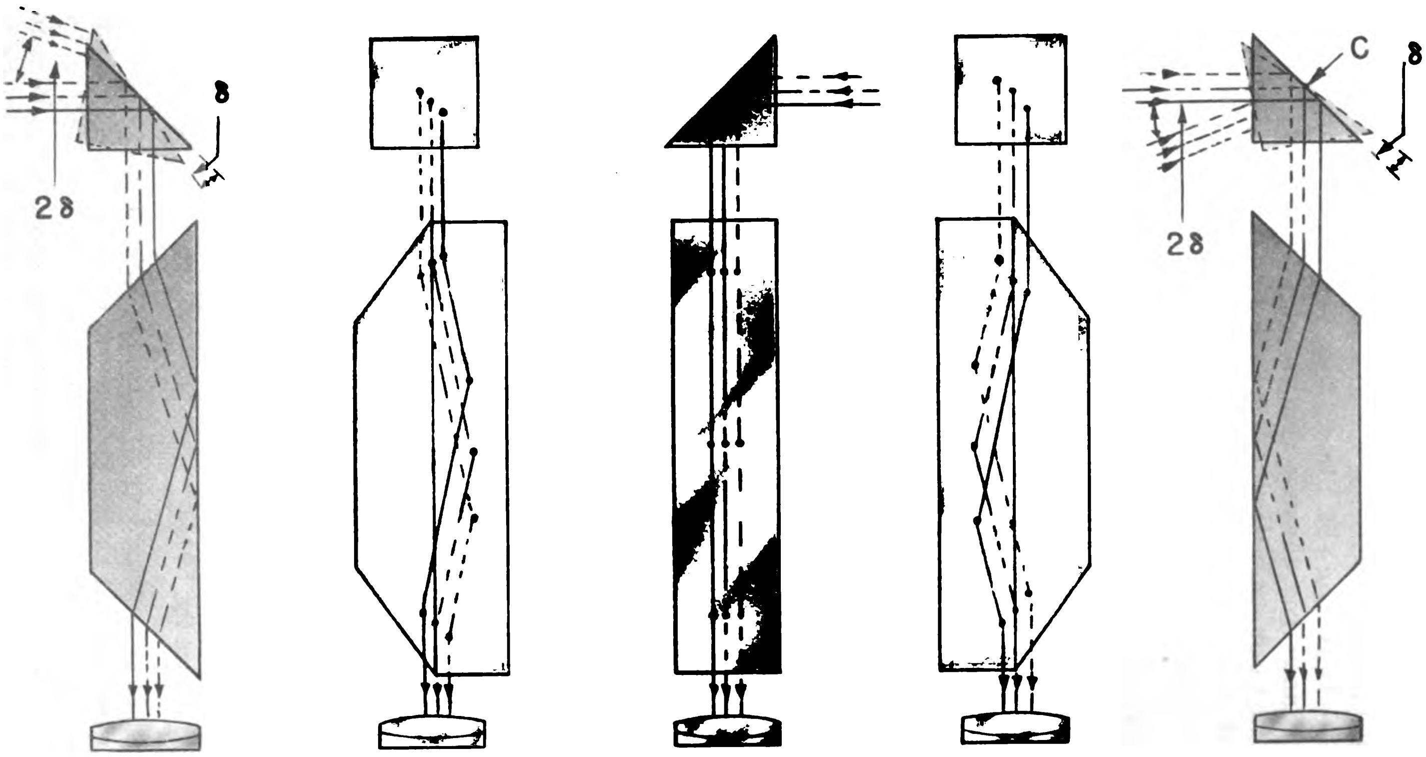


Figure 57. The rotating prism system, type I, by C. P. Goerz.

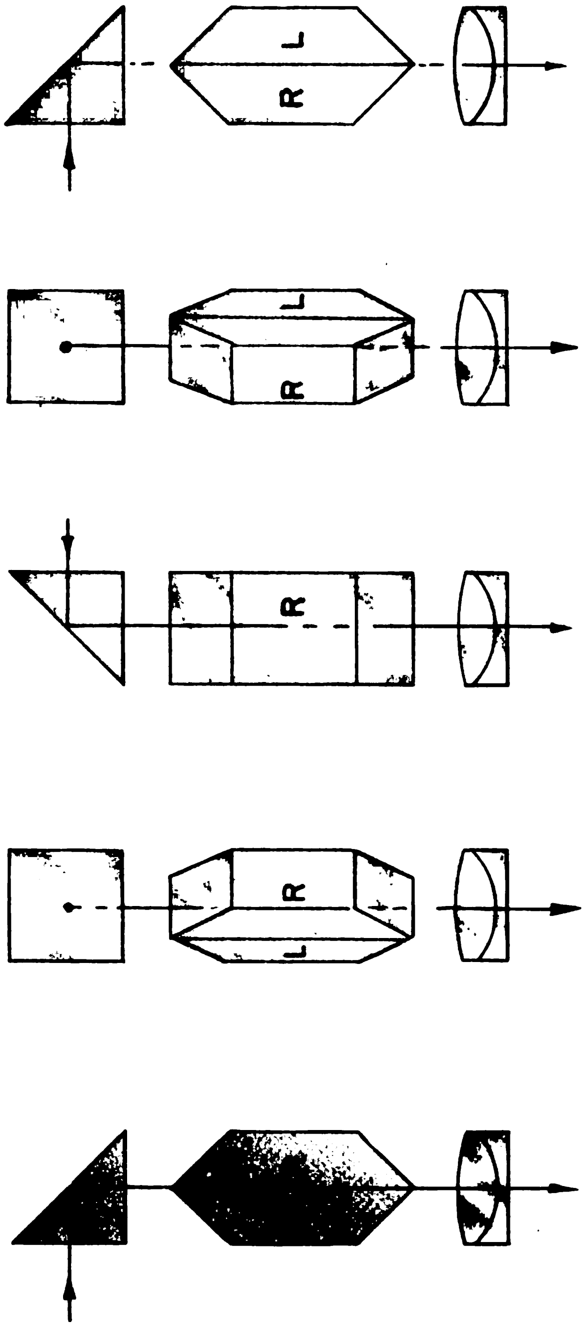


Figure 58. The rotating prism system by Carl Zeiss.

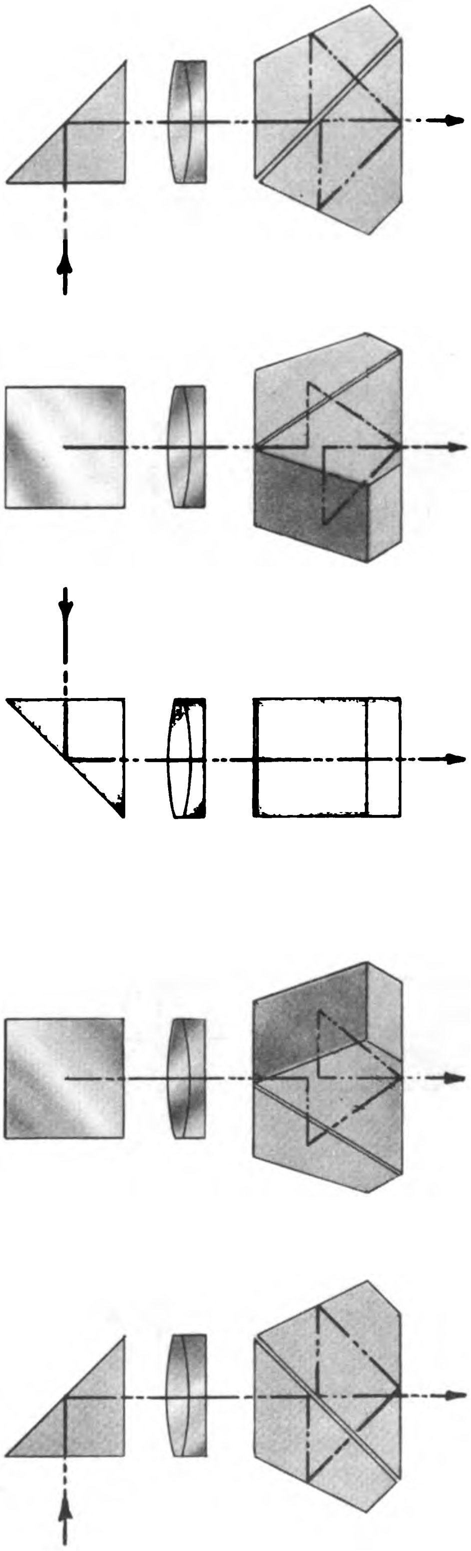


Figure 59. Rotating prism system, type II, by C. P. Goerz.

viewing different parts of the horizon, the two prisms are so geared that the right-angle prism is turned through an angle 2α when the second prism is rotated through the angle α. This is illustrated in figures 57 to 59.

In order to increase the true field of view of the instrument in the vertical plane, the right-angle prism is mounted so that it may be elevated or depressed about the axis C. This axis is perpendicular to the plane of the paper. When this prism is elevated or depressed, the angles of incidence on the reflecting surface of the prism become smaller than the critical angle, and for this reason it becomes necessary to silver the reflecting surface of the right-angle prism.

44. Special Prisms

In this paragraph some prisms will be discussed which are not actually used in fire-control instruments, but which are employed in instruments used in testing optical components or the material of which these components are made.

The Abbe Constant Deviation Prisms. Two of these prisms are shown in figure 60; they are used in spectrometers. Each of these prisms is made in one piece. The first prism (a) may be considered built up of two 30° prisms and a right-angle reflecting prism, as indicated by the dash-lines in diagram (a). The refracting angles of the prism are as follows:

$$\theta_1 = 75^\circ \qquad \theta_2 = 135^\circ \qquad \theta_3 = 60^\circ$$

The incident ray will be refracted and dispersed at the entrance surface of the prism and, proceeding, it will be totally reflected by the second surface toward the exit surface. Here it will again be refracted so that the incident ray and the emerging ray include an angle of 90°, provided the sine of the angle of incidence satisfies the equation:

$$\sin i_1 = n_D \sin 30^\circ \tag{107}$$

The second prism, diagram (b), has the following refracting angles,

$$\theta_1 = 90^\circ \qquad \theta_2 = 60^\circ \qquad \theta_3 = 30^\circ$$

In this case the deviation is equal to 60° if equation (107) is fulfilled. As a rule, the deviation required is for the D-line of the spectrum. In order to obtain the desired deviation, that is, in order that equation (107) is satisfied, we turn the prism through a small angle about the axis C (perpendicular to the plane of the paper) until the desired result is obtained.

Example:

The two prisms (a) and (b) just described are made of a dense flint glass which has the following constants:

$$n_C = 1.74835 \qquad n_D = 1.7561 \qquad n_F = 1.77613$$

The angle of incidence is the same for both prisms and its magnitude is determined with equation (107); hence, $i_1 = 61^\circ 24' 28.8''$ and, therefore ${}_{C}i_1' = 30^\circ 8' 48.1''$, ${}_{D}i_1' = 30^\circ$, and ${}_{F}i_1' = 29^\circ 37' 39.4''$.

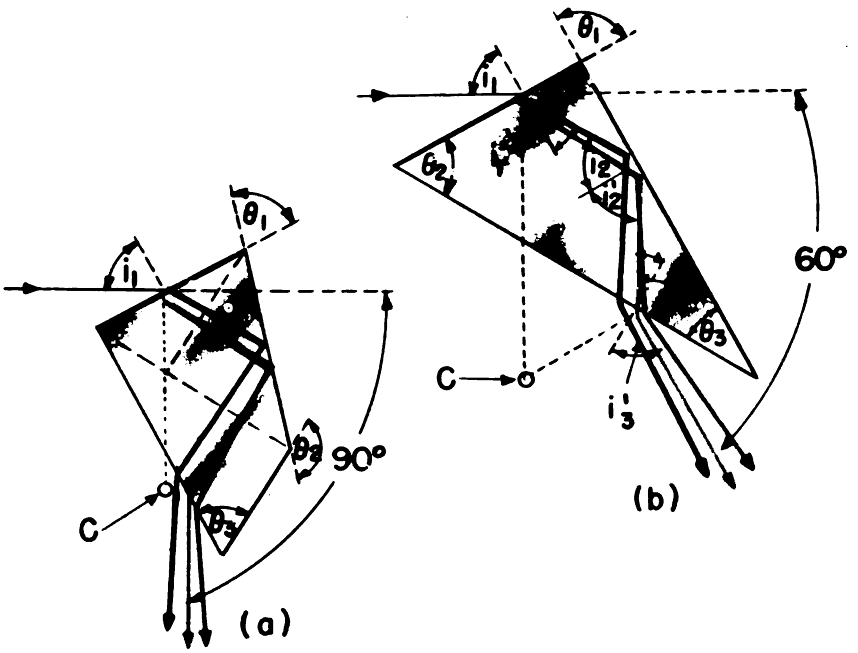


Figure 60. Abbe's dispersion prisms.

Now we apply the following equations:

$$i_2 = -i_2' = i_1' - \theta_1 \qquad i_3 = i_2' - (\theta_1 - \theta_3) \text{ in case (a), and } i_3 = i_2' - \theta_3 \text{ in case (b)}$$

$$\Delta = i_1 - i_3' + 90° \text{ and } \Gamma = \Delta_C - \Delta_F = {}_Fi_3' - {}_Ci_3'$$

Substituting the known values into these equations, we have:

Prism (a)

$\lambda =$	C	D	F
$i_2 = -i_2' =$	$-44°51'11.9''$	$-45°$	$-45°22'20.6''$
$i_3 =$	$+29°51'11.9''$	$+30°$	$+30°22'20.6''$
$i_3' =$	$+60°29'34.8''$	$+61°24'28.8''$	$+63°53'\ 7.2''$
$\Delta =$	$90°54'54''$	$90°$	$87°30'21.6''$

$$\Gamma = 3°24'32.4''$$

Prism (b)

All the constants are the same as above except the values for i_2 which are in this case:

$\lambda =$	C	D	F
$i_2 = -i_2' =$	$-59°51'11.9''$	$-60°$	$-60°22'20.6''$

The Amici Dispersion Prism. The Amici dispersion prism, shown in figure 61(a), is used in connection with the spectroscope, and also with the Abbe refractometer. It is made up of three single prisms which are cemented together with Canada balsam. The center prism is made of a dense flint glass, and the outer two prisms of borosilicate crown glass. The magnitudes of the refracting angles are computed so that there will be no deviation for light corresponding to the D-line of the spectrum, that is, $\Delta_D = 0$; hence, the prism is a direct vision prism.

We proceed to determine the various constants of the prism, considering the light for the D-line only. The glasses used have the following constants:

Prism =	n_C	n_D	n_F
1 & 3 =	1.51109	1.5136	1.51931
2 =	1.74835	1.7561	1.77613

The refracting angle θ_2 is chosen at will, and we put, $\theta_2 = -72°$. Then apply the following equations:

$$i_3 = -i_2' = -\frac{\theta_2}{2} = +36°$$

$$\sin i_3' = -\sin i_2 = \frac{n_2 \sin i_3}{n_3} = 0.681956 \qquad i_3' = -i_2 = 42°59'48.1''$$

$$\tan \theta_1 = \tan \theta_3 = \frac{n_3 \sin i_3' + \sin \frac{\theta_2}{2}}{n_3 \cos i_3' - \cos \frac{\theta_2}{2}} = 1.491259$$

$$\theta_1 = \theta_3 = +56°9'18.7''$$

$$i_4 = -i_1' = i_3' - \theta_1 = -13°9'30.6''$$

$$\sin i_4' = -\sin i_1 = n_3 \sin i_4 = -0.344565 \qquad i_4' = -i_1 = -20°9'18.7''$$

$$\Delta = i_4' + \theta_3 + \frac{\theta_2}{2} = 0$$

$$\alpha = 90° - \left(\theta_1 + \frac{\theta_2}{2}\right) = 69°50'41.3''$$

$$\beta = 90° + \frac{\theta_2}{2} = 54°$$

$$A = 11 \text{ mm (chosen at will)}$$

$$B = -2\ A \tan \frac{\theta_2}{2} = 15.98 \text{ mm}$$

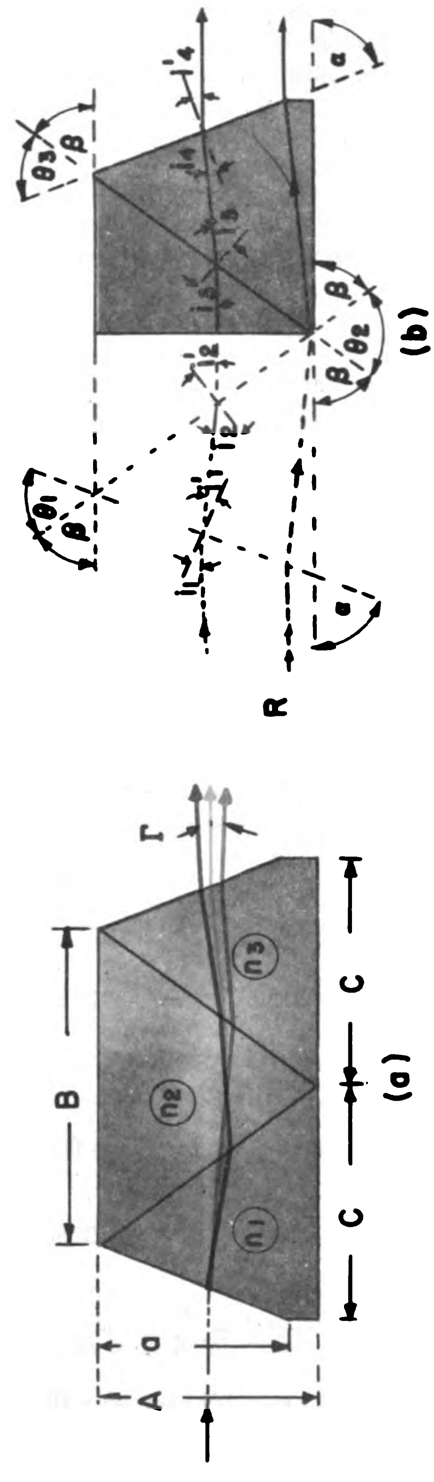


Figure 61. The Amici dispersion prism.

$$a = \frac{A \cos i_1 \cos i_2}{\cos i_1 \cos i_2 + \sin \theta_1 \sin (i_1 - i_1')} = 9.59 \text{ mm}$$

$$C = a \tan i_1 - A \tan \frac{\theta_2}{2} = 11.51 \text{ mm}$$

In order to determine the dispersion Γ caused by the prism, we trace the rays through the system with the aid of the following equations:

$$\sin i_v' = \frac{n_{v-1} \sin i_v}{n_v}$$

$$i_v = i_{v-1}' - \theta_{v-1}$$

$$\Delta = i_4' + \left(\frac{\theta_2}{2} + \theta_3 \right)$$

$$\Gamma = \Delta_C - \Delta_F$$

$\lambda =$	C	D	F
$\sin i_1 =$	0.344565	0.344565	0.344565
$\sin i_1' =$	0.228024	0.227646	0.226790
$i_1 =$	20° 9′18.7″	20° 9′18.7″	20° 9′18.7″
$i_1' =$	13°10′50.9″	13° 9′30.6″	13° 6′29.5″
$i_2 =$	— 42°58′27.8″	— 42°59′48.1″	— 43° 2′49.2″
$\sin i_2 =$	— 0.681672	— 0.681956	— 0.682598
$\sin i_2' =$	— 0.589165	— 0.587785	— 0.583897
$i_2' =$	— 36° 5′50.6″	— 36° 0′ 0″	— 35°43′30.4″
$i_3 =$	+ 35°54′ 7.9″	+ 36° 0′ 0″	+ 36°16′29.6″
$\sin i_3 =$	0.586402	0.587785	0.591660
$\sin i_3' =$	0.678474	0.681956	0.691672
$i_3' =$	+ 42°43′28.2″	+ 42°59′48.1″	+ 43°45′45.4″
$i_4 =$	— 13°25′50.5″	— 13° 9′30.6″	— 12°23′33.4″
$\sin i_4 =$	— 0.232269	— 0.227646	— 0.214609
$\sin i_4' =$	— 0.350979	— 0.344565	— 0.326058
$i_4' =$	— 20°32′49.9″	— 20° 9′18.7″	— 19° 1′46.9″
$\Delta =$	— 0°23′31.2″	± 0° 0′ 0″	+ 1° 7′31.8″

Γ = — 1°31′3.0″

If a greater dispersion Γ is desired, substitute a greater value for θ_2 and repeat the computations.

The Rutherford Dispersion Prism. This prism is used in the so-called prism circle of the spectroscope introduced by Kirchhoff and Bunsen. The combination consists of a flint glass prism cemented between two crown glass prisms as illustrated in figure 62(a).

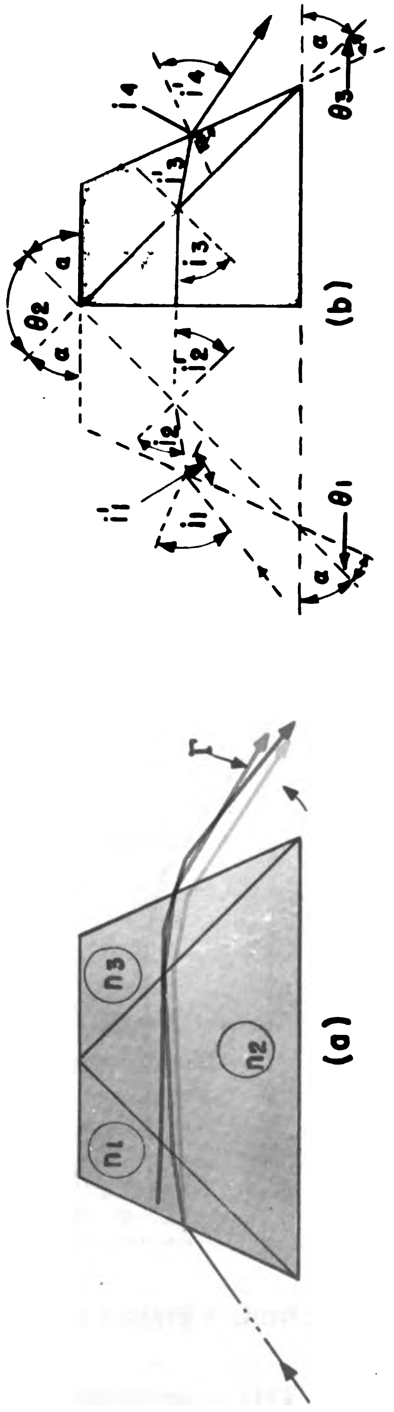


Figure 62. The Rutherford dispersion prism.

This prism system yields a much greater dispersion than the Amici prism. The prism is placed on the spectroscope table so that it will produce its minimum deviation. In that case, the light for the D-line will pass symmetrically through the prism and, therefore, when computing the various angles of the prism, we consider only the rear half of the system which is drawn in full lines in figure 62(b).

We will determine here the angles of the prism, assuming that the prisms are made of the same glasses used in the Amici prism. The refracting angle θ_2 of the flint glass prism must be made large enough so that no light can pass through the prism when the latter is in air, that is, the angle θ_2 must be at least twice as great as the critical angle J for that particular glass. In our case $J = 34°42'41''$ and, hence, θ_2 must not be less than $69°25'22''$. Computing the angles, we consider light for the D-line only, and we also select the following values at will: $\theta_2 = +90°$ and $\Delta_D = -69°$. With the formulae given in the preceding paragraph we find:

$$\alpha = i_3 = -i_2' = -\frac{\theta_2}{2} = -45°$$

$$i_3' = -i_2 = -55°7'27.8''$$

$$\tan\theta_1 = \tan\theta_3 = \frac{n_3 \sin i_3' + \sin\frac{\theta_2 - \Delta_D}{2}}{n_3 \cos i_3' - \cos\frac{\theta_2 - \Delta_D}{2}} = -0.378339$$

$$\theta_1 = \theta_3 = -20°43'25.0''$$

$$i_4 = -i_1' = -34°24'2.9''$$

$$i_4' = i_1 = -58°46'35.0''$$

Applying equations (79) to (82) we find,

$$\Delta_C = -66°14'54'' \quad \Delta_D = -69° \quad \text{and} \quad \Delta_F = -76°57'59.0'' \quad \text{hence}$$

$$\Gamma = \frac{\Delta_C - \Delta_D}{2} = +5°21'31.7''$$

The factor of one half enters the calculation of Γ since we have considered only half the prism in the above calculation. It can be seen that the total dispersion would be just half that due to one half the prism because of its symmetry.

The Hüfner Prism. This prism is made in one piece as shown in figure 63(a). If used in ordinary light it will produce two spectra, one of them in reversed order. However, the prism is used extensively with colorimeters when comparing the light transmissions from two sources of light. A parallel beam of light coming from the filter F_1 is lowered by transmission through the prism, while the beam of light from filter F_2 is raised. Upon emerging from the prism the two beams will travel parallel to each other but the right hand edge of the prism will divide the field into two equal parts by a sharp

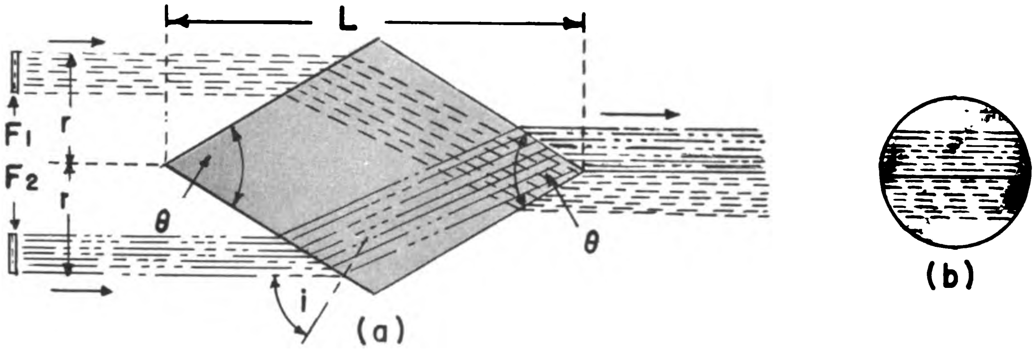


Figure 63. The Hüfner prism.

boundary line as illustrated in figure 63(b). If r is the distance from the axis to the extreme ray which is permitted to pass through the prism, and n is the refractive index of the glass, then the prism must have a length of

$$L = r\left\{\text{cotan}\,\frac{\theta}{2} + \tan\left(\frac{\theta}{2} + i'\right)\right\}$$

hence, if $\theta = 60°$, $n = 1.5165$, and $r = 0.875$ inch, we have:

$$i' = 90° - \frac{\theta}{2} = 60°$$

$$\sin i' = \frac{\sin i}{n} = 0.5710685 \qquad i' = 34°49'29.3''$$

$$L = 3.359 \text{ inches}$$

The Lummer-Brodhun Prism. This cube, shown in figure 64(a), is used in connection with photometers. It consists of the two right-angle reflecting prisms P_1 and P_2. If the hypothenuse surfaces are brought into contact the prisms will form a cube. The reflecting surface of the prism P_1 is ground in a concave grinding tool until only a small circular portion of the polished surface remains. The polished surface is then applied to the reflecting surface of prism P_2 under such a pressure that the clean surfaces adhere in optical contact. The central portion of each beam of light will now pass freely through the faces in contact, but the outer portions of the beam incident from the right will be totally reflected. The outer portions of the beam incident from the left will be absorbed by the ground section of the spherical surface. If a collective lens is placed in the path of the emergent light, the image formed by this lens will consist of two concentric circles as illustrated in figure 64(b). The smaller circle is formed by the light which has passed through both prisms, while the larger circle is formed by light reflected by the prism P_2.

The Bi-Prism. Bi-prisms are used in photometers, binocular microscopes, and other scientific instruments, for the purpose of dividing a beam of light. They are called biprisms because they may be considered as being built up of two prisms placed base to

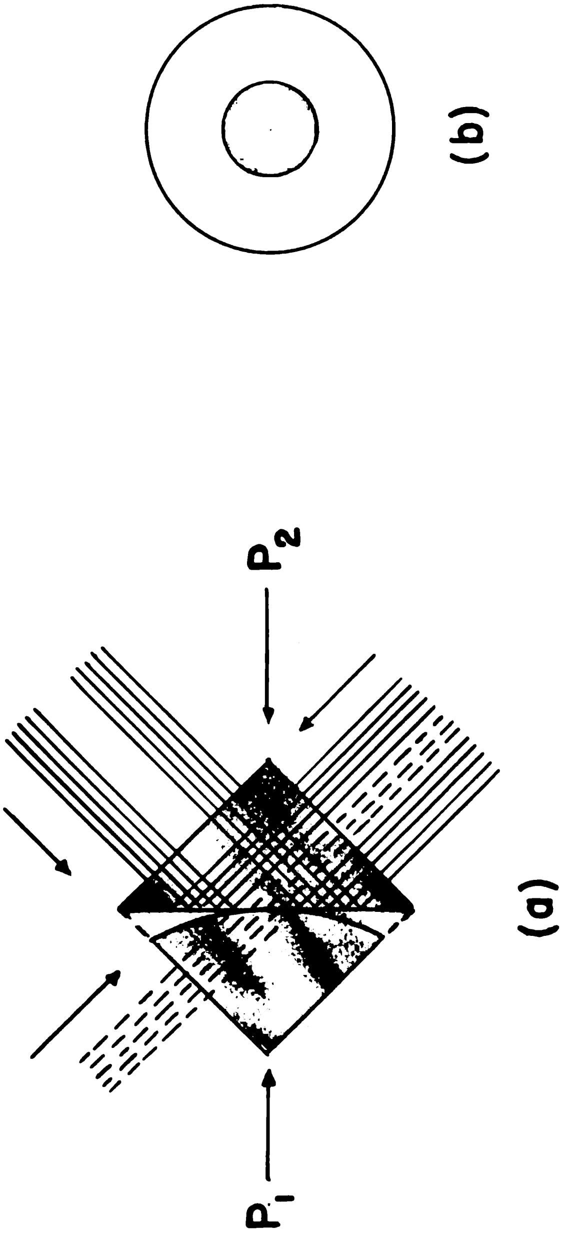


Figure 64. The Lummer–Brodhun cube.

base. In figure 65(a) there is shown the Fresnel bi-prism whose refracting angles are small. If the angular deviation of each prism is given, we apply equation (101) in order to determine the magnitude of the refractive angle θ. Binocular microscopes employ achromatic bi-prisms, as, for example, the Wenham prism illustrated in figure 65(b). The magnitude of the refracting angles are determined so that equations (102) and (103) are satisfied.

The Nicol Prism. The Iceland spar crystal ABCD is shown in its natural form in figure 66(a). The angles included by the faces AD and BC and the edges AB and CD, respectively, are equal to 71°. The first step in converting the crystal into a Nicol prism is to grind and polish the end faces so that the refracting angle θ_1 (see fig. 66(b)) will be equal to 68°. In the first diagram the new end surfaces are indicated by the dash-lines AD and B′C. The next step consists in cutting the prism diagonally from B′ to D′ so that the new surfaces are normal to the end faces. After these surfaces are ground, carefully polished, and cleaned they are cemented together with Canada balsam. In figure 66(b) is shown a principal section of the finished Nicol prism. The final refracting angles are,

$$\theta_1 = 68^\circ \qquad \theta_2 = 22^\circ \qquad \text{and} \qquad \theta_3 = 90^\circ$$

Due to the double-refracting characteristic of the Iceland spar, the incident ray will be divided into two rays at the point of incidence, namely, into the "*ordinary*" ray and into the "*extraordinary*" ray. For the ordinary ray the index of refraction is $n = 1.6585$, while for the extraordinary ray, $n = 1.4865$. The refractive index of the Canada balsam is, $n = 1.546$. The ordinary ray, indicated by the dotted line in figure 66(b), will be totally reflected by the cemented surface B′D′ and is then absorbed at the face AB′. The extraordinary ray will travel through both halves of the prism and then emerge from the end face B′C and proceed parallel to the incident ray. The waves in the emerging light will vibrate only in one plane now, that is, the light has been polarized. The Nicol prism is used in polariscopes and photometric instruments.

45. Range Measuring Compensators

The range measuring conpensators consist of one or two wedge-shaped prisms. These wedges may be either a single prism or a pair of achromatic prisms. Compensator wedges are used in range finders to bring into coincidence the images formed by the two objectives. The observer will read the range on the range scale, the range being determined by the amount the wedges had to be displaced, or by the angle through which they had to be rotated in order to obtain coincidence of the two images. Two methods are employed in securing coincidence of the images by means of wedges, namely:

Method I: The wedge, which is located between the objective and its image plane, is moved along the optical axis until the images are in coincidence. In this case the wedge is referred to as the "*sliding range measuring wedge.*"

Method II: Here two wedges are used which are turned through equal angles in opposite senses until the two images are in coincidence. In this case wedges are known as the "*rotating range measuring wedges.*"

46. Sliding Range Measuring Wedge

The sliding wedge is mounted between the objective and the focal plane of the same. Its actions will be more readily understood by reference to figure 67. Let Δ be the angu-

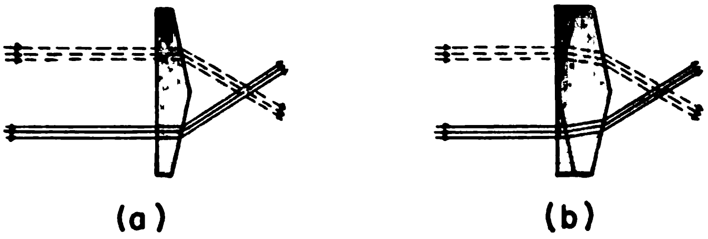

(a) (b)

Figure 65. The Bi-prism.

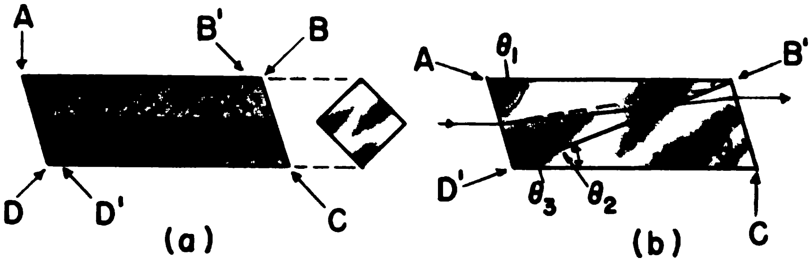


Figure 66. The Nicol prism.

lar deviation caused by the wedge and θ the refracting angle of the wedge. The existing relations between these two constants are expressed by equations (99) and (100), and f is the focal length of the objective.

Rays of light entering the objective O parallel to the optical axis will come to a focus at the point c in the focal plane I. If we place the wedge in the path of the rays at a distance E in front of the image plane, the image point c will move to the position indicated by the point c′ as illustrated in figure 67(a). If D is the displacement of the point c, then,

$$D = cc' = E\Delta \tag{108}$$

This position of the wedge will be referred to as the *"infinity position"* of the sliding wedge. Rays of light proceeding from targets at closer ranges will form the angle ε with axis, and the image point c will be displaced as shown in figure 67(b). In order to bring this point at the point c′ we must move the sliding wedge laterally from its infinity position a distance equal to L. We have

$$D = cc' = dd' = f\varepsilon = L\Delta \tag{109}$$

If B is the base-length of the instrument and R is the range to the target, then,

$$\varepsilon = \frac{B}{R} \tag{110}$$

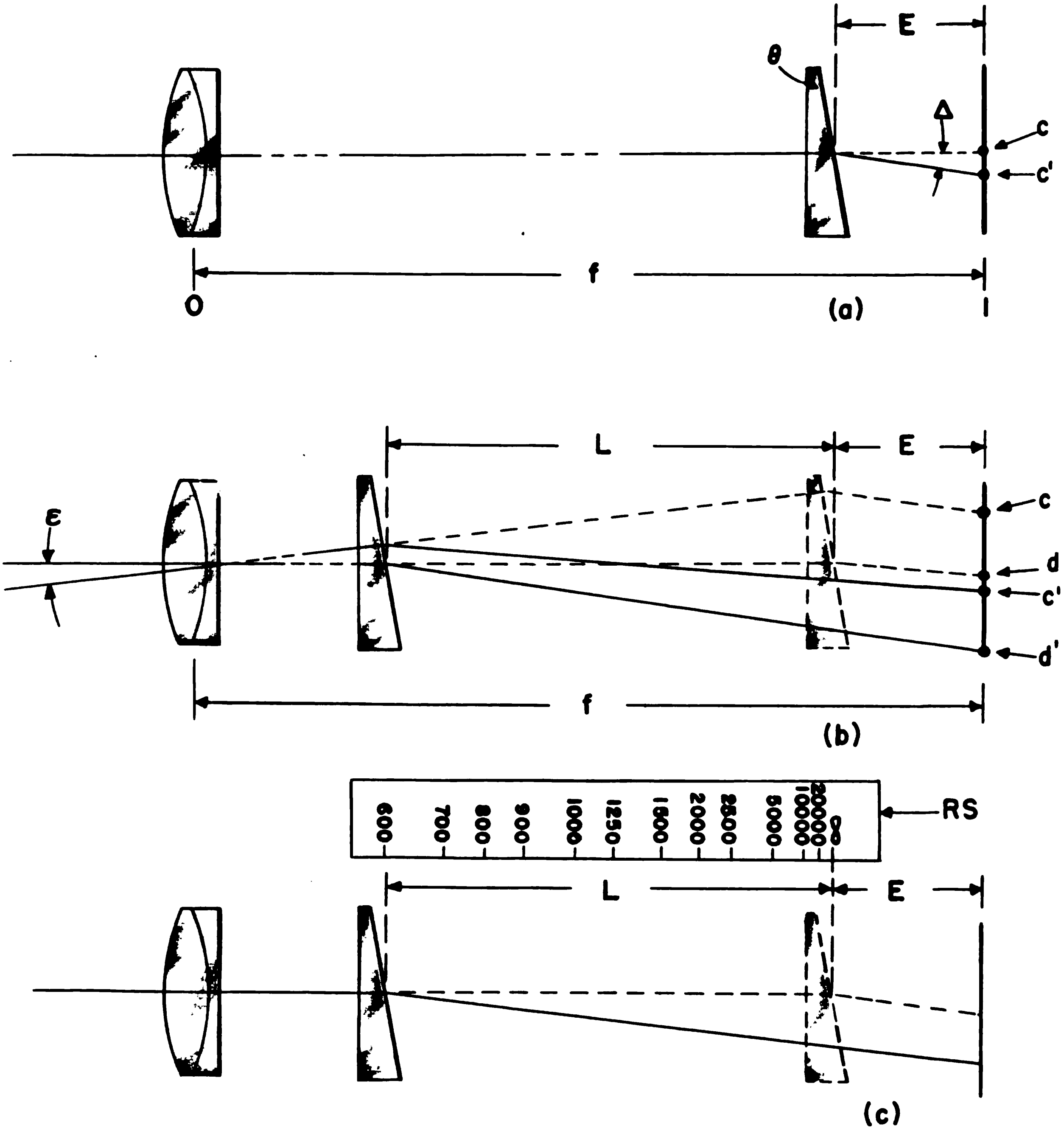


Figure 67. The range compensator by Barr & Stroud.

Substituting equation (110) into (109) we obtain the following equation for the range,

$$L = \frac{Bf}{R\Delta} \tag{111}$$

where Δ is expressed in radians. With the aid of this equation we are able to determine the magnitude of L for any given range. The distances L are indicated by graduated lines on the range scale RS (see fig. 67), each line being marked with the corresponding range either in meters or yards. The range scale is mounted so that the index will point to ∞ when the wedge is in its infinity position as in figure 67(a). The range is determined with the following equation:

$$R = \frac{Bf}{L\Delta} \tag{112}$$

Differentiating this equation in respect to f, Δ, and R, we get:

$$\delta R = \frac{R^2\ L\Delta}{Bf^2}\ \delta f \tag{113}$$

$$\delta R = -\ \frac{R^2\ L}{Bf}\ \delta\Delta \quad \text{and,} \tag{114}$$

$$\delta\Delta = -\ \frac{Bf}{R^2\ L}\ \delta R \tag{115}$$

Example:

A range finder has the following constants:

$B = 1$ meter $= 1.09361$ yards $\qquad f = 8.85$ inches ± 1/2%

$\Delta = 1°3'40''$

If we substitute these values into equation (111), considering a range of 10,000 yards, that is, $R = 10{,}000$, we find $L = 0.05226$ inches.

The tolerance in f is equal to ± 1/2%, hence, $\delta = 0.044$ inch. Substituting this value into equation (113) we find, $\delta R = \pm 49.7$ yards, that is, an error of ± 1/2% of the range R. A glance at equation (112) will at once reveal the fact that an error in R due to variations in f can be neutralized by selecting a sliding wedge whose deviation varies accordingly, that is, the error $\delta\Delta$ must be large enough to offset the error in δR. In our example we found $\delta R = \pm 49.7$ yards. Substituting this value into equation (115), we find $\delta\Delta = \pm 19''$. Hence, we either change the graduations on the scale (due to the variation in f) or we pair an objective with a sliding wedge so that in each case the correct range will be read. Below are tabulated three such combinations where, in the second example, it has been assumed that f and Δ have their true computed values:

(1) $f = 8.806$ $(\delta f = -\ 0.044)$ with $\Delta = 1°3'21''$ $(\delta\Delta = -\ 19'')$

(2) $f = 8.850$ $(\delta f = \pm\ 0)$ with $\Delta = 1°3'40''$ $(\delta\Delta = +\ 0)$

(3) $f = 8.894$ $(\delta f = +\ 0.044)$ with $\Delta = 1°3'59''$ $(\delta\Delta = +\ 19'')$

47. Rotating Range Measuring Wedge

The rotating wedges are always used in pairs and are placed in front of the objective. This is an advantage over the sliding wedge type since the focal length of the objective does not affect the accuracy of range reading. The action of this compensator will be easily understood by reference to figure 68. The wedges are circular in shape but for clearness they are shown in rectangular section in the diagrams.

In the diagrams (a) and (b) the wedges are shown in positions for producing maximum deviation. In these positions the deviation is equal to 2Δ, where Δ is the deviation caused by one wedge. If the wedges are turned through equal angles β in opposite senses (see diagram (c)), the deviation of the compensator will then be equal to $\pm 2\Delta \cos \beta$. Hence, the change in deviation from the maximum deviation produced by this rotation is equal to $2\Delta\ (1 - \cos \beta)$. Continue to rotate the wedges until the angle $\beta = 90°$ is reached, a case illustrated in diagram (d), then $2\Delta\ (1 - \cos \beta) = 2\Delta$, that is, there will be no deviation and the wedges will act as a plane-parallel plate. The magnitude of the parallactic angle ε is given by the equation (110) $\varepsilon = \frac{B}{R}$, but:

$$\varepsilon = 2\Delta\ (1 - \cos \beta) \quad \text{hence,} \tag{116}$$

$$\frac{B}{R} = 2\Delta\ (1 - \cos \beta) \quad \text{and, therefore,} \tag{117}$$

$$R = \frac{B}{2\Delta\ (1 - \cos \beta)} \tag{118}$$

Here it has been assumed that $\beta = 0$ and in this case the angle will be designated by β_∞ If β varies from 0 then equation (118) must be written thus,

$$R = \frac{B}{2\Delta} \left\{ \frac{1}{\cos \beta_\infty - \cos \beta} \right\} \tag{119}$$

where Δ is expressed in radians. Differentiating this equation with regard to 2Δ we have,

$$\delta R = - \frac{R^2\ (\cos \beta_\infty - \cos \beta)}{B}\ \delta 2\Delta \tag{120}$$

The magnitude of the angle β for any given range R can be computed with the aid of the following equation:

$$\cos \beta = \cos \beta_\infty - \frac{B}{R 2\Delta} \tag{121}$$

Example:

A range finder has the following known constants:

$B = 1\text{m} = 1.09361$ yds. $\qquad 2\Delta = 3'37.06'' \qquad \beta_\infty = 0$

How accurate must the deviation 2Δ be if the ranges are to be read with the same accuracy as the one obtained in the sliding wedge type discussed in the preceding paragraph?

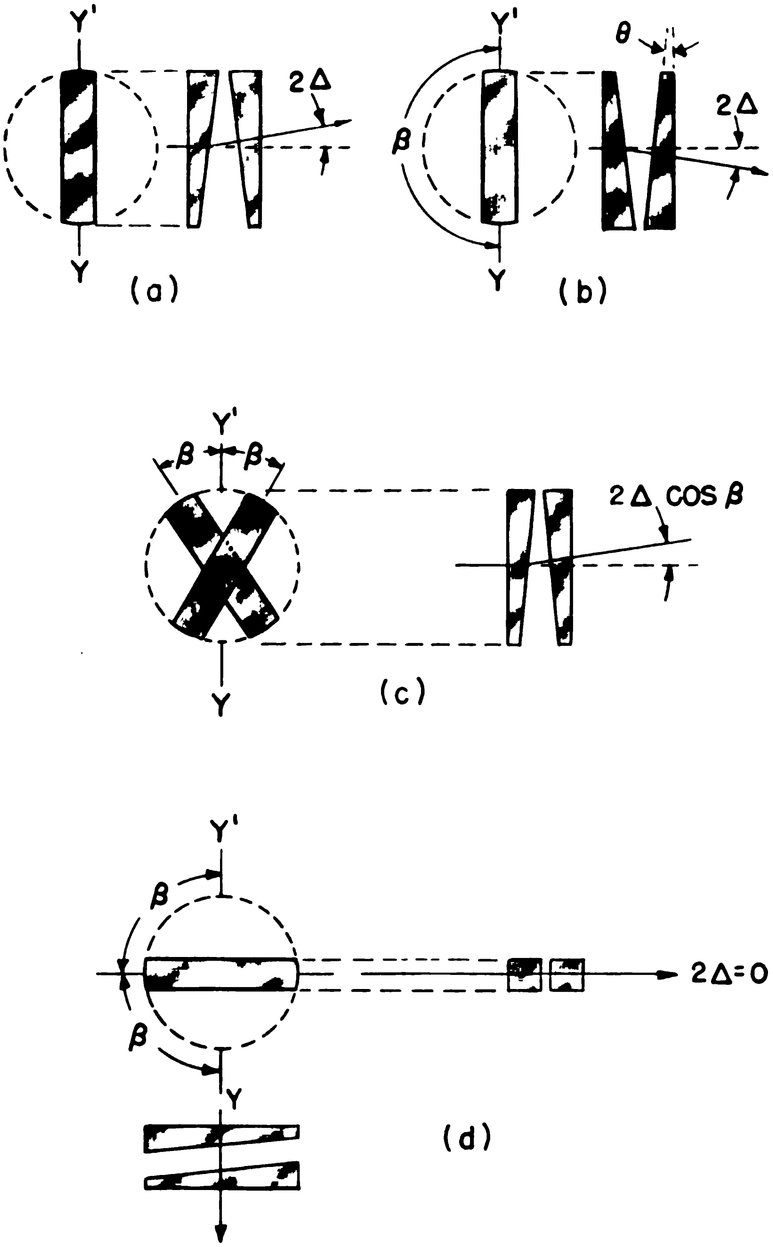


Figure 68. The range compensator by Carl Zeiss.

For a range of 10,000 we found an error of $\delta R = \pm 49.7$ yards. In order to determine the value of $\delta 2\Delta$ which will yield the same accuracy, we rearrange equation (120) thus,

$$\delta 2\Delta = -\frac{B}{R^2 (\cos \beta_\infty - \cos \beta)} \delta R \qquad (122)$$

and, by substituting for δR the value of 49.7 yards we find,

$$\delta 2\Delta = \pm 1.08''$$

Let us consider the uncertainty of observation with range finders. Assuming the optical components of the instrument to be perfect, there still will be an error in the range reading corresponding to an angle ω. This error represents the smallest angle that can be appreciated when using the range finder. Its magnitude depends not only on the limiting angle of resolution of the observer's eye but it is also affected by other influences such as the nature of the target, the illumination, etc. The error δR due this angular error ω is determined as follows:

$$\delta R = \pm \frac{R^2 \omega}{BM} \qquad (123)$$

where M is the magnification of the range finder.

Assuming the resolving power in free vision is 10 seconds of arc for one observer and 15 seconds of arc for a second observer, then the possible error in reading the range at 10,000 yards under particularly favorable conditions, using a one meter base range finder of magnification $M = 15X$, will be equal to ± 295.5 yards for the first observer and ± 443.3 yards for the second one. No matter how conscientious the observer may be, this uncertainty remains.

There are three methods by which we can obtain a greater accuracy in range reading, namely,

(1) Increase the magnification M of the range finder.

(2) Increase the base-length B of the instrument, and,

(3) Select an observer whose eyes can resolve in free vision angles of 10 seconds of arc or less.

König and Martin have found observers whose eyes could resolve from 3 to 10 seconds of arc after a little practice, or even better in special cases. For instance, with a 3 meter base range finder whose magnification is equal to 28X and an observer whose resolving power in free vision is equal to 5 seconds of arc, the resulting error in reading the range at 10,000 yards will be reduced to $\delta R = \pm 26$ yards.

Returning to our rotating range measuring wedges for the 1 meter base range finder it seems advisable to demand an accuracy in range reading of ± 1%, hence, applying equation (122) we find, $\delta 2\Delta = \pm 2$ seconds of arc, that is, the tolerance in deviation for each wedge must not exceed the limit of ± 1 second of arc.

When pairing the wedges select two of equal deviations so that the total deviation of the combination is equal to $3'37.06'' \pm 2''$ when the wedges are in a position as indicated in diagrams (a) or (b) in figure 68. This deviation may also be obtained by matching wedges of unequal deviations but in that case the image formed by the objective directly behind the compensator will move in a slanting direction instead of a horizontal direction, when the wedges are rotated. This type of range compensator is employed by Carl Zeiss.

48. Measuring Constants of a Prism

For measuring accurately the refracting angles of a prism there is employed an instrument called the "*spectrometer.*" A plan view of the essential parts of this instrument is shown in figure 69. The collimator contains the objective O_1 and in its focal plane the slit S, which is illuminated by the discharge tube T. The observing telescope houses the objective O_2, the eye piece E, and the cross-hairs C.H. in the focal plane of the objective. The collimator is in a fixed position, but the telescope can be rotated about the vertical axis of the instrument. The prism-table P.T., upon which is placed the prism to be tested, can also be rotated about the same axis. Degrees and minutes of arc are read on the graduated circle C, and seconds of arc are read with the vernier V.

The prism P whose refracting angle θ is to be measured is placed on the prism-table so that its edge A will about bisect the beam of light coming from the collimator as illustrated in figure 70. Turn the telescope until the image of illuminated slit, which is formed by the light reflected by the prism surface A—B, appears on the intersection of the cross hairs. The telescope is now in the position indicated by dash-lines. Take a reading on circle and vernier to determine the exact position of the telescope. Then turn the telescope until it will receive the light reflected by the prism surface A—C, and bring the image of the slit in coincidence with the intersection of the cross-hairs. Take a second reading of the telescope position. The magnitude of the refracting angle θ is equal to the difference in the two readings divided by two. For example, if the first reading obtained is equal to 180°30′45″, and the second reading is 60°30′45″, then,

$$\theta = \frac{180°30'45'' - 60°30'45''}{2} = 60°$$

In order to measure the deviation Δ caused by a prism proceed as follows. Position the telescope so that its axis will coincide with the axis of the collimator. The image of the slit will then appear on the intersection of the cross-hairs. Take a reading of the telescope position. Next place the prism, whose refracting angle θ has already been determined, on the prism-table in the approximate position indicated by the dash-lines in figure 71. Turn both the table with prism and the telescope in the direction of the arrow until the image of the slit again appears on the intersection of the cross-hairs. Continue to turn the table with prism and the telescope in the same direction, keeping the image of the slit on the intersection of the cross-hairs, until the point is reached where the image of the slit will begin to move in the opposite direction. At that moment we have reached the point of minimum deviation for the prism. Now take a second reading of the telescope position. The minimum deviation caused by the prism is equal to the difference in the two readings. For example,

first reading at 210°30′45″, and,

second reading at 171°19′41″, then,

$$\Delta = 210°30'45'' - 171°19'41'' = 39°11'4''$$

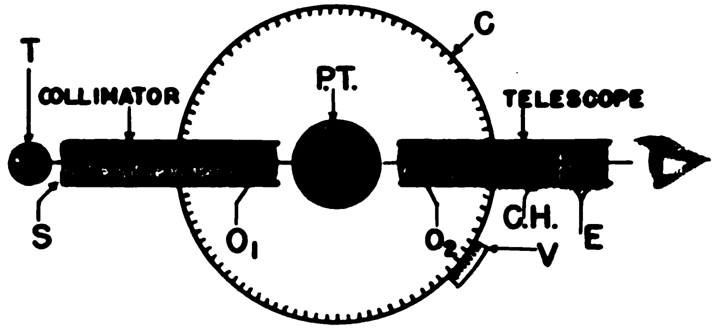


Figure 69. Plan view of the spectrometer.

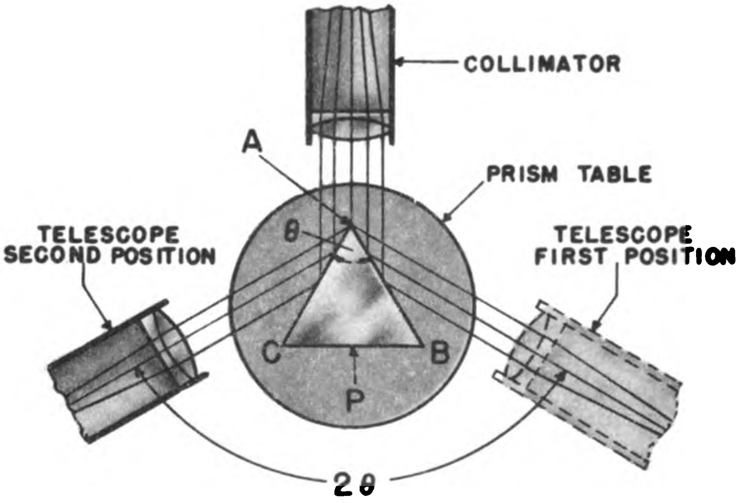


Figure 70. Measuring the refracting angle θ of the prism.

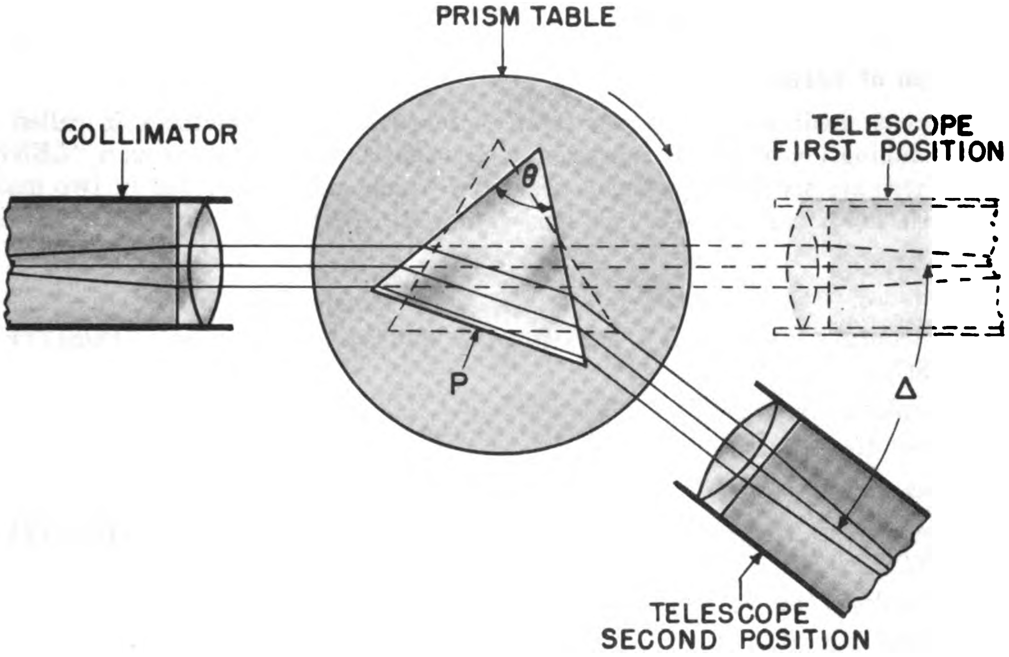


Figure 71. Measuring the minimum deviation of a prism.

Knowing the values of Δ and θ we are able to determine the refractive index of the prism glass with equation (81).

$$n = \frac{\sin \frac{\theta + \Delta}{2}}{\sin \frac{\theta}{2}} = \frac{\sin 49°35'32''}{\sin 30°} = 1.5229$$

SECTION V

LENSES AND LENS-SYSTEMS

PART 1. INTRODUCTION

49. Classification of Lenses

A transparent medium bounded by two centered spherical surfaces is called a *"LENS."* A combination of two or more lenses used as a unit is known as a *"LENS-SYSTEM."* There are six different forms of lenses which may be divided in two main groups, *"COLLECTIVE"* or *"POSITIVE"* lenses, and *"DISPERSIVE"* or *"NEGATIVE"* lenses.

Collective lenses may be:

- *a.* Convex-concave (fig. 72(a)). This form of lens is also known as a *"POSITIVE MENISCUS LENS."*
- *b.* Convex-plane (fig. 72(b)).
- *c.* Bi-convex (fig. 72(c)).

Dispersive lenses may be:

- *a.* Concave-convex (fig. 72(d)). This form of lens is also called a *"NEGATIVE MENISCUS LENS."*
- *b.* Concave-plane (fig. 72(e)).
- *c.* Bi-concave (fig. 72(f)).

If the absolute values of the two radii of a bi-convex or bi-concave lens are equal, we speak of *"EQUI BI-CONVEX"* or *"EQUI BI-CONCAVE"* lenses respectively. Two such lenses are illustrated in figure 72(g) and 72(h).

We will deal only with spherical surfaces unless otherwise stated. Furthermore, we will restrict our investigation to rays of light which travel very close to the axis, that is, in a space called the "paraxial region." We may, therefore, assume that all rays proceeding from a point-object will also form a point-image and, hence spherical aberration is assumed to be eliminated.

For many practical purposes the thickness of the lens may be neglected and in that case we speak of a *"thin lens."* We assume, therefore, that the two spherical refracting surfaces and the two nodal points (see definition following) of the lens coincide at the common vertex.

50. Relation Between Lenses and Prisms

We have seen earlier how to determine the path of a ray through either a lens or a prism when certain information is available. For a lens we need the refractive index, radii, and thickness and for a prism, the refractive index and refracting angle.

It will be shown here that the refracting power of a plane-convex lens is equal to that of a single prism whose refracting angle θ is equal to the angle of incidence i_2 at the spherical surface of the lens. The following condition must be satisfied:

$$\sin\,\theta = -\sin\,i_2 = -\frac{h}{R_2} \qquad (124)$$

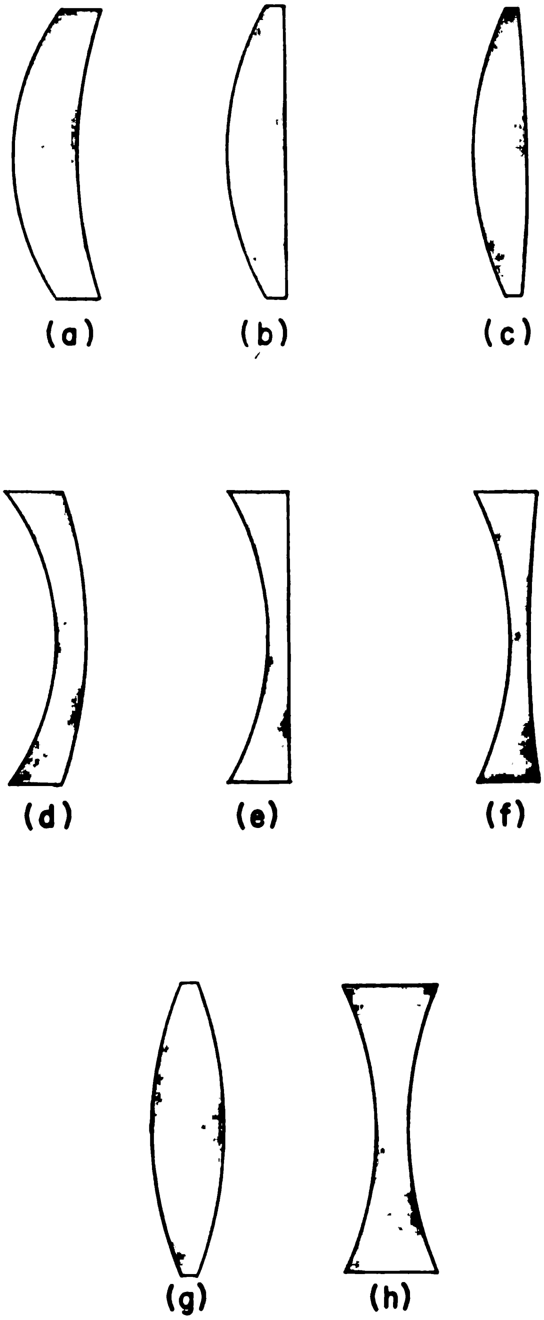


Figure 72. Various forms of lenses.

where h is the distance from the axis to the incident ray coming from a distant object as illustrated in figure 73. The plane-convex lens shown in the diagrams has the following constants:

$$R_1 = \pm \infty \qquad d = 0.875$$

$$R_2 = -1.5165 \qquad n = 1.5165$$

C is the center of curvature of the spherical surface and ω is the angle included by the axis and the radius drawn through the point of incidence at the spherical surface.

In order to construct the prism, which will have the same refracting power as the plane-convex lens, proceed as follows:

Let the first surface of the lens be also the first face of the prism. Draw a normal to the radius through the point of incidence at the spherical surface. This line is the second face of the prism, and the angle included by the two prism faces is $\theta = -i_2$. From the diagrams in figure 73 we derive easily

$$\sin i_2 = -\sin \theta = \sin \omega = \frac{h}{R_2} \tag{125}$$

Case 1:

$$h = +0.50$$

$$\sin i_2 = -\sin \theta = \sin \omega = -0.3297065 \quad \text{or,}$$

$$i_2 = -\theta = -19.25° = -19°15'$$

$$\sin i_2 = n \sin i_2 = -0.50 \quad \text{or} \quad i_2' = -30°$$

Case 2:

$$h = +1.00 \text{ inch}$$

$$\sin i_2 = -\sin \theta = \sin \omega = -0.6594131 \quad \text{or,}$$

$$i_2 = -\theta = \omega = -41.255° = -41°15'12''$$

However, this is equivalent to the critical angle J for this glass (see equation (7a)) and, hence, $i_2' = -90°$, that is, we have a case of grazing emergence as illustrated in figure 73(b).

Case 3:

$$h = +1.25 \text{ inch}$$

$$\sin i_2 = -\sin \theta = \sin \omega = -0.8242664 \quad \text{or,}$$

$$i_2 = -\theta = \omega = -55.514° = -55°30'50''$$

In this case the angle i_2 is greater than the critical angle J and the ray is prevented from entering into air, that is, the ray is totally reflected back so that $i_2' = -i_2$, as illustrated in figure 73(c).

A spherical surface may be thought of as being made up of an infinite number of small plane surfaces as illustrated in figure 74. The small central plane face is parallel to the plane surface of the lens while the other small planes include angles θ with the front surface. The greater the distance between the axis and the small plane surfaces the greater the angle θ. The rays marked a, b, c, d, and e proceed from a distant object and will pass undeviated through the plane surface of the lens and arrive at the spherical surface. The first four rays will be refracted there and enter into air where

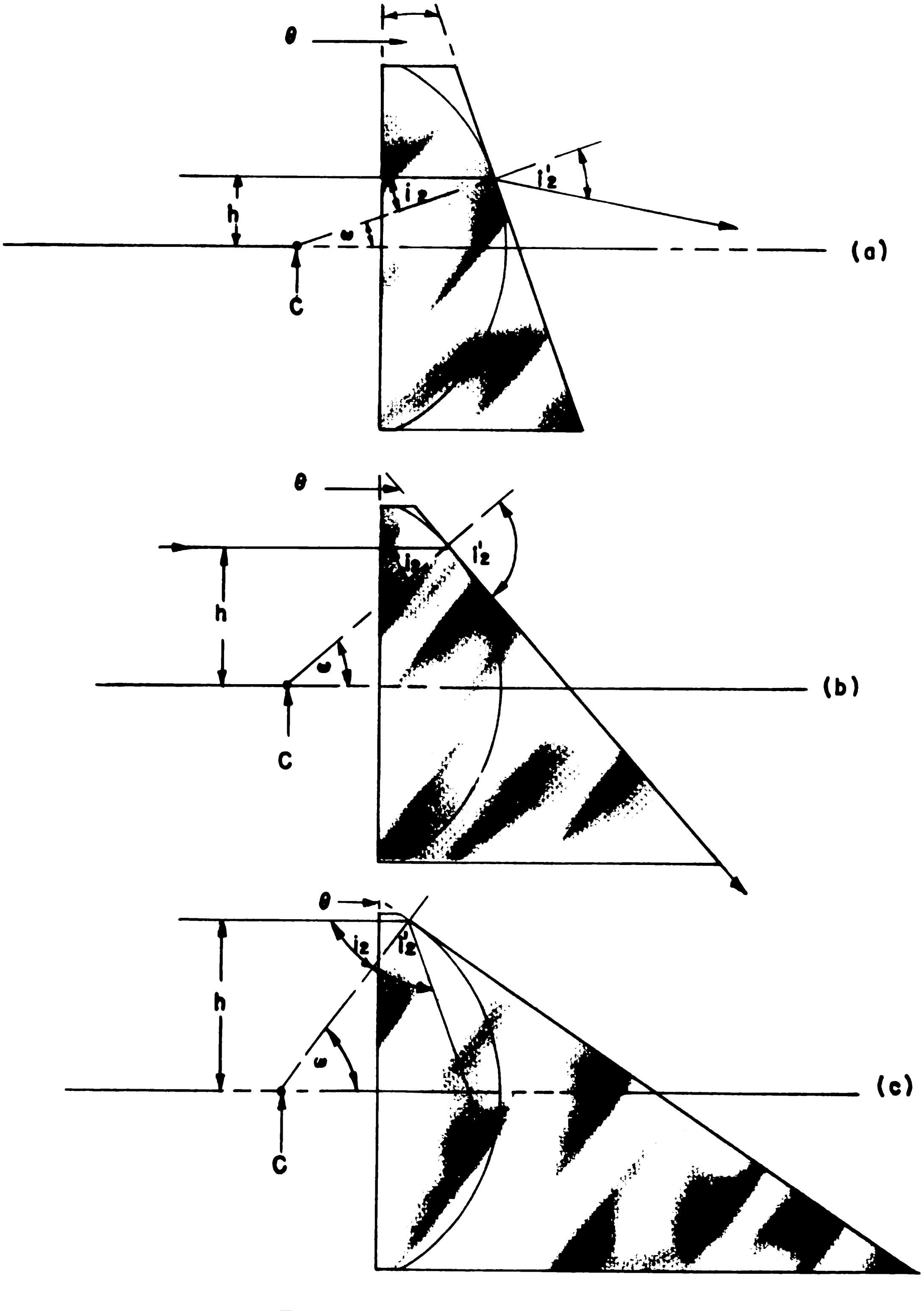


Figure 73. Plane-convex lens and prism.

they will intersect the axis at the points designated by the symbols a′, b′, c′, and d′, respectively. The ray e is totally reflected back into the lens and will intersect the axis at the point e′. The distance a′—e′ is termed the *"SPHERICAL ABERRATION."* However, only rays intercepting the axis between a′ and d′ contribute to the image formed by the lens so this interval is measured as the spherical aberration of the lens.

The bi-convex lens shown in figure 75 has the following known constants:

$R_1 = +1.500$ inches $\quad d = 1.100$ inches $\quad n = 1.5165$

$R_2 = -2.000$ inches $\quad h = 0.750$ inches

From the Law of Refraction, and from figure 75 we derive

$$\sin i_1 = \frac{h}{R_1} = +0.500 \qquad \text{hence, } i_1 = +30^\circ$$

$$\sin i_1' = \frac{\sin i_1}{n} = +0.3297065 \qquad \text{hence, } i_1' = +19.251^\circ = 19^\circ 15' 4''$$

$$s_2 - R_2 = \frac{R_1 \sin i_1'}{\sin (i_1 - i_1')} + (R_1 - R_2 - d) = +5.304161$$

$$\sin i_2 = \frac{(s_2 - R_2) \sin (i_1 - i_1')}{R_2} = -0.4516320, \text{ hence, } i_2 = -26.8484^\circ = -26^\circ 50' 54''$$

$$\sin i_2' = n \sin i_2 = -0.6848999, \text{ hence, } i_2' = -43.2277^\circ = -43^\circ 13' 40''$$

$$\omega_1 = i_1 = 30^\circ \qquad \omega_2 = i_1' - (i_1 + i_2) = +16.0994^\circ = +16^\circ 5' 58''$$

A prism, made of the same glass, will have the same refracting power as the lens when its angles are of the following magnitudes,

$$\theta = \omega_1 + \omega_2 = 46^\circ 5' 58'' \qquad \alpha = 90^\circ - \omega_1 = 60^\circ$$

$$\beta = 90^\circ - \omega_2 = 73^\circ 54' 2''$$

51. Notations and Sign Conventions

B.F.L. = the back focal length of the lens, or the distance from the rear surface of the lens to the second focal point F (fig. 76).

C_v = the center of curvature of the vth surface (fig. 82).

d = the distance between the two surfaces of a lens, measured along the axis, or the thickness of the lens (fig. 81).

D = the distance between two lenses (fig. 86). In the case of thick lenses it is measured from the second Gauss point of the first lens to the first Gauss point of the second lens.

f = the equivalent focal length of the lens, or the distance from the second Gauss point of the lens to the second focal point F (fig. 76). It is considered positive when the point F lies to the right of the lens and vice versa.

f′ = the first focal distance, or the distance from the first Gauss point of the lens to the first focal point F′ (fig. 76). It is considered negative when the point F′ is to the left of the lens and vice versa.

F' = the first focal point of the lens (fig. 76).

F = the second focal point of the lens (fig. 76).

F.F.L. = the front focal length of the lens, or the distance from the vertex of the front surface to the first focal point F' (fig. 76(d)).

H = the distance from the vertex of the front surface of the lens to the first Gauss point P (fig. 79). It is positive when the point lies to the right of the front surface and vice versa.

H' = the distance from the rear surface of the lens to the second Gauss point P' (fig. 79). It is negative when the point P' lies to the left of the vertex and vice versa.

H'' = the distance between the two Gauss points.

K = the distance from the front vertex to the first nodal point N of the lens. It is positive when N is to the right of the vertex and vice versa (fig. 80).

K' = the distance from the rear surface to the second nodal point N' (fig. 80). It is considered negative when N' lies to the left of the vertex and vice versa.

K'' = the distance between the two nodal points.

M = the optical center of the lens (fig. 81).

m_1 = the distance from the vertex of the front surface of the lens to the optical center M (fig. 81).

m_2 = the distance from the vertex of the rear surface of the lens to the optical center (fig. 81).

n_o = the refractive index of the medium in front of the lens (fig. 78).

n_1 = the refractive index of the medium between the two refracting surfaces (fig. 78).

n_2 = the refractive index of the medium behind the lens (fig. 78).

N = the first nodal point of the lens (fig. 80).

N' = the second nodal point of the lens (fig. 80).

P = the first Gauss point, or the first principal point of the lens (fig. 79).

P' = the second Gauss point, or the second principal point of the lens (fig. 79).

R_v = the radius of curvature of the vth surface. It is considered positive when the surface is convex towards the incident light and vice versa.

s = the object distance, that is, the distance from the first Gauss point to the object (target) T (fig. 77). It is negative when the object is to the left of the Gauss point and vice versa.

s' = the image distance, that is, the distance from the second Gauss point to the image (fig. 77). It is positive when the image is to the right of the second Gauss point and vice versa.

T = an object point (fig. 77).

T′ = an image point (fig. 77).

x = the distance from the first focal point F′ of the lens to the object. It is considered negative when the object is to the left of the point F′ (fig. 82) and vice versa.

x′ = the distance from the second focal point F to the image (fig. 82). It is positive when the image lies to the right of F and vice versa.

y = the height of the object measured from the axis (fig. 78). It is positive when measured above the axis.

y′ = the height of the image measured from the axis (fig. 78). It is positive when measured above the axis.

α = the angle included by the axis and a ray in object-space (fig. 78).

α' = the angle included by the axis and a ray in image-space (fig. 78).

β = the lateral magnification. It is defined by equation (128).

γ = the angular ratio. It is defined by equation (127).

μ = the angle included by the axis and the principal ray in object-space (fig. 78).

μ' = the angle included by the axis and the principal ray in image-space (fig. 78).

ρ_v = the reciprocal value of R_v.

σ and σ' = the reciprocal values of s and s′, respectively.

ϕ = the reciprocal value of f, or the power of the lens.

52. Focal Lengths—Focal Planes—Focal Points

In figure 76(a) are illustrated the light paths of the two rays designated by the letters a and b, respectively, through a spherical surface which separates glass from air. The ray a proceeds from a distant target in object space, that is, it travels parallel to the optical axis. Upon passing through this spherical surface the ray will be refracted and then, proceeding in image space, it will intersect the axis at the point F. This point is called the *"SECOND FOCAL POINT"* of the refracting surface. A plane through this point, normal to the axis, is known as the *"SECOND FOCAL PLANE,"* designated by the letters SFP in the diagrams. The distance f from the vertex of the surface to SFP is termed the *"SECOND FOCAL DISTANCE"* or the *"SECOND FOCAL LENGTH."* The ray b, proceeding from a distant point in image-space, travels parallel to the axis while traveling in glass. After refraction by the surface it will emerge into air and intersect the axis at the point F′. This point is called the *"FIRST FOCAL POINT"* and the plane drawn through this point, normal to the axis, is the *"FIRST FOCAL PLANE,"* designated by the letters FFP. The distance f′ from the vertex of the spherical surface to the point F′ is known as the *"FIRST FOCAL LENGTH"* of the refracting surface.

In figure 76(b) and (c) are shown the light paths of the same rays through a thin collective lens and through a thin dispersive lens, respectively. In the case of the dis-

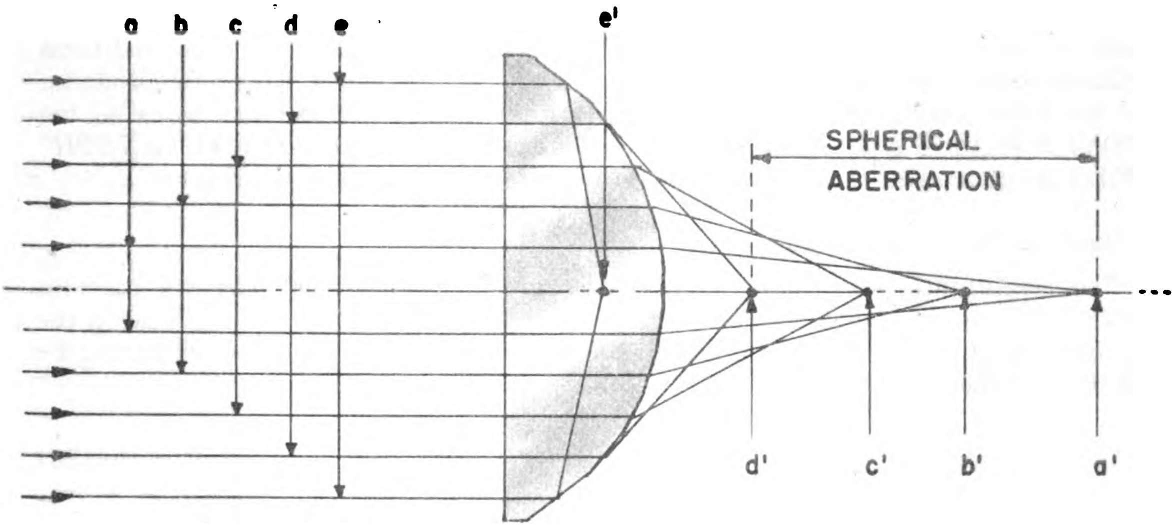


Figure 74. The spherical aberration.

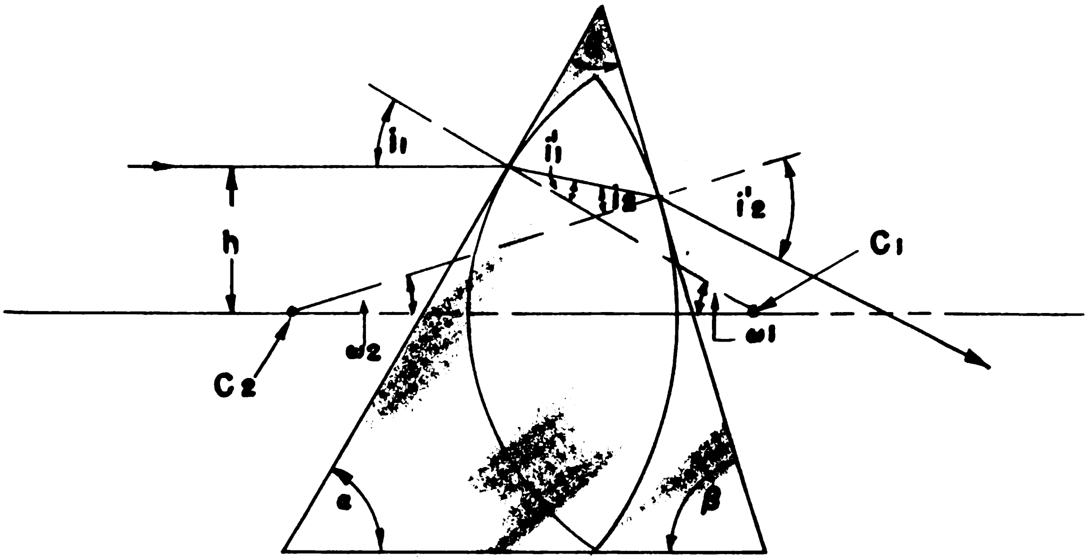


Figure 75. The bi-convex lens and the prism.

persive lens the positions of f, f′, F, F′, and SFP and FFP are reversed due to the fact that the emerging ray, produced backwards, determines the positions of these lengths, points, and planes. In the case of a thick lens the distances f and f′ are measured from the Gauss points (or principal points) of the lens as shown in figure 76(d). The distance from the front vertex of the lens to the first focal point F′ of the lens is called the *"FRONT FOCAL LENGTH"* (F.F.L.) of the lens, while the *"BACK FOCAL LENGTH"* (B.F.L.) is equal to the distance from the rear surface of the lens to the point F.

53. Dioptric Power of a Lens

The magnifying power of a lens becomes greater as its focal length f becomes shorter. Its power is inversely proportional to the focal length. For this reason the reciprocal value of f is called the *"POWER OF THE LENS"* and it is designated by the letter D, hence

$$D = \phi = \frac{1}{f} \tag{126}$$

If f is expressed in meters, m, and if we put $f = 1\text{m}$, then the power of the lens is $D = 1$. This unit of power is called a *"DIOPTER"* (D_p). Thin lenses, such as spectacle lenses having focal lengths of 2, 3, and 4 meters, have dioptric powers of 1/2, 1/3 and 1/4 diopters, respectively.

Example 1:

A lens has a focal length of 250 millimeters. What is its dioptric power?

$$f = 250 \text{ mm} = 0.25 \text{ m, hence,}$$

$$D = \frac{1}{0.25} = 4 \ D_p$$

Example 2:

What is the dioptric power of a lens whose focal length is equal to − 10 inches?

$$f = -10 \text{ inches} = -0.254 \text{ meters, therefore,}$$

$$D = -\frac{1}{0.254} = -3.94 \ D_p$$

If the power of a lens is given in diopters we find the focal length (in inches) as follows: Divide 1000 by the number of diopters multiplied by 25.4, or, if f is to be expressed in millimeters divide 1000 by the number of diopters.

Example 3:

What is the focal length of a lens whose dioptric power is equal to − 2.5 diopters?

$$f = -\frac{1000}{2.5 \times 25.4} = -15.75 \text{ inches, or, } f = -\frac{1000}{2.5} = -400 \text{ mm}$$

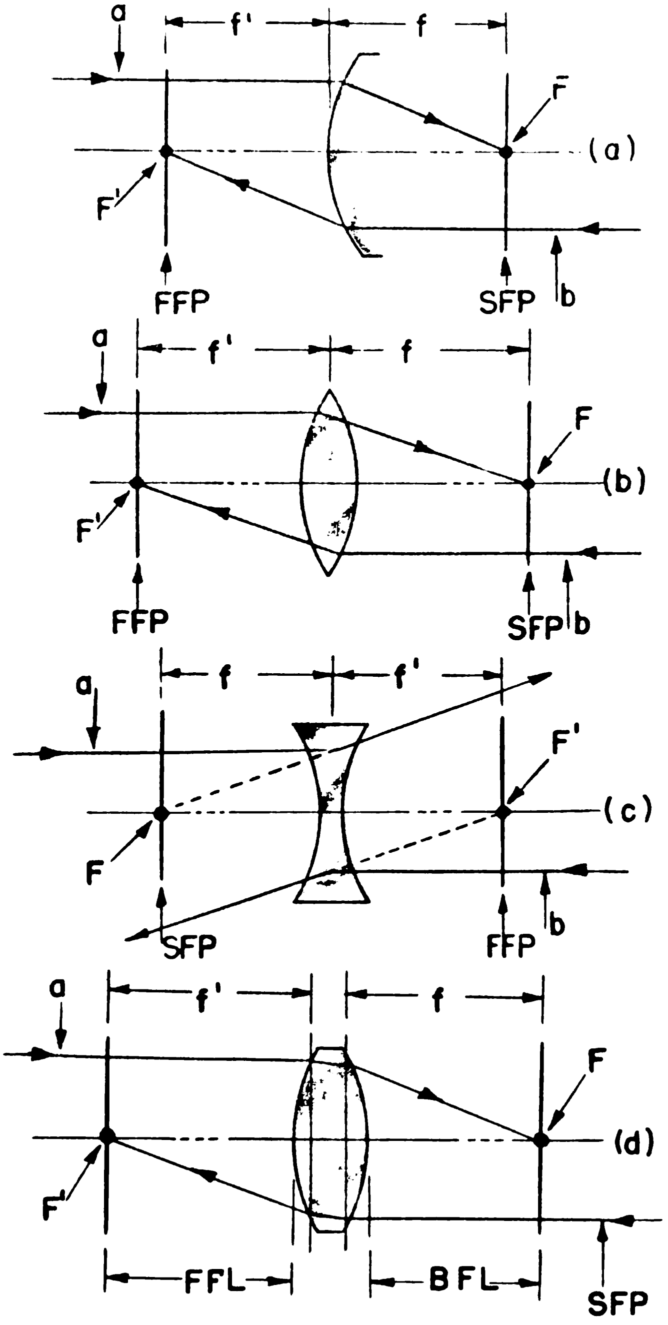


Figure 76. Focal lengths—focal planes—focal points.

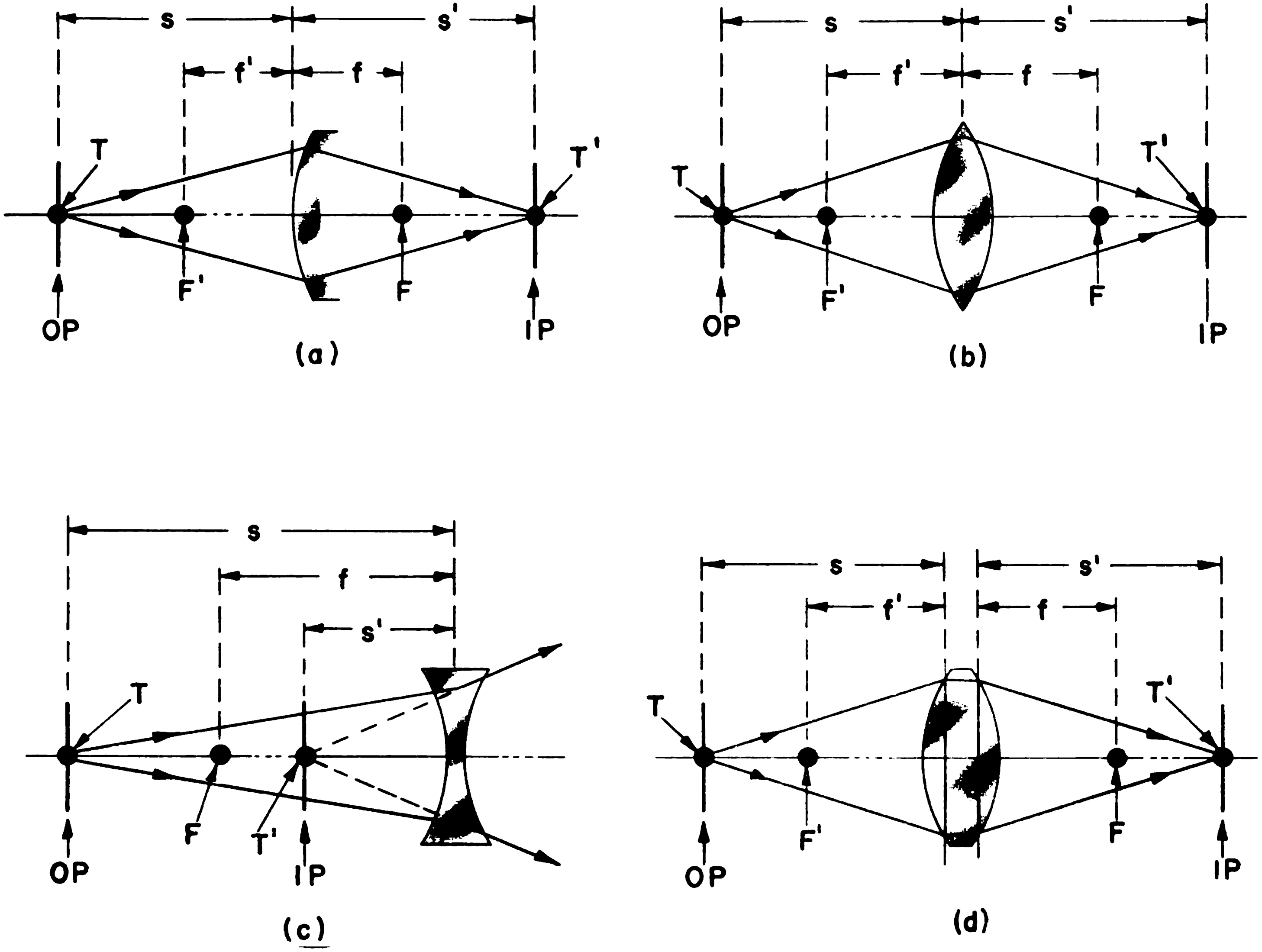


Figure 77. Object and image planes—conjugate planes and points.

In table XI are tabulated the dioptric powers of lenses and their equivalent focal length in inches.

TABLE XI

DIOPTERS	INCHES	DIOPTERS	INCHES	DIOPTERS	INCHES	DIOPTERS	INCHES
0.25	157.480	5.25	7.499	10.50	3.750	20.50	1.920
0.50	78.740	5.50	7.158	11.00	3.579	21.00	1.875
0.75	52.493	5.75	6.847	11.50	3.423	21.50	1.831
1.00	39.370	6.00	6.562	12.00	3.281	22.00	1.790
1.25	31.496	6.25	6.299	12.50	3.150	22.50	1.750
1.50	26.247	6.50	6.057	13.00	3.028	23.00	1.712
1.75	22.497	6.75	5.833	13.50	2.916	23.50	1.675
2.00	19.685	7.00	5.624	14.00	2.812	24.00	1.640
2.25	17.498	7.25	5.430	14.50	2.715	24.50	1.607
2.50	15.748	7.50	5.249	15.00	2.625	25.00	1.575
2.75	14.316	7.75	5.080	15.50	2.540	25.50	1.544
3.00	13.123	8.00	4.921	16.00	2.461	26.00	1.514
3.25	12.114	8.25	4.772	16.50	2.386	26.50	1.486
3.50	11.249	8.50	4.632	17.00	2.316	27.00	1.458
3.75	10.499	8.75	4.499	17.50	2.250	27.50	1.432
4.00	9.843	9.00	4.374	18.00	2.187	28.00	1.406
4.25	9.264	9.25	4.256	18.50	2.128	28.50	1.381
4.50	8.749	9.50	4.144	19.00	2.072	29.00	1.358
4.75	8.288	9.75	4.038	19.50	2.019	29.50	1.335
5.00	7.874	10.00	3.937	20.00	1.969	30.00	1.312

54. Object and Image Planes—Conjugate Planes and Points

In the diagrams of figure 77 the object points T are shown at a finite distance s from the first refracting surface. The paraxial ray, proceeding from T will be refracted by the spherical surface (diagram (a)), or by the lenses shown in diagrams (b), (c), and (d) and then it will proceed in image space. It will finally intersect the axis at the point T′. T′ is the image point of the object point T; hence, T′ is the *"CONJUGATE POINT"* to T. A plane through T, normal to the axis, is termed the *"OBJECT PLANE"* (OP) and a plane through T′ is the *"IMAGE PLANE"* (IP). The image plane IP is the conjugate plane to the object plane OP. s is known as the *"OBJECT DISTANCE"* and s′ is the *"IMAGE DISTANCE."*

55. Angular Ratio—Lateral Magnification

The angular ratio γ of the two conjugate points T and T′ shown in figure 78 is equal to the quotient of the two angles of inclination in image space and object space. This is mathematically expressed as follows:

$$\gamma = \frac{\alpha'}{\alpha} \tag{127}$$

The lateral magnification β in the conjugate planes $T - T_1$ and $T' - T_1'$ is equal to the size of the image divided by the size of the object, which is equal to the quotient of the image distance and the object distance. Hence,

$$\beta = \frac{y'}{y} = \frac{s'}{s} \tag{128}$$

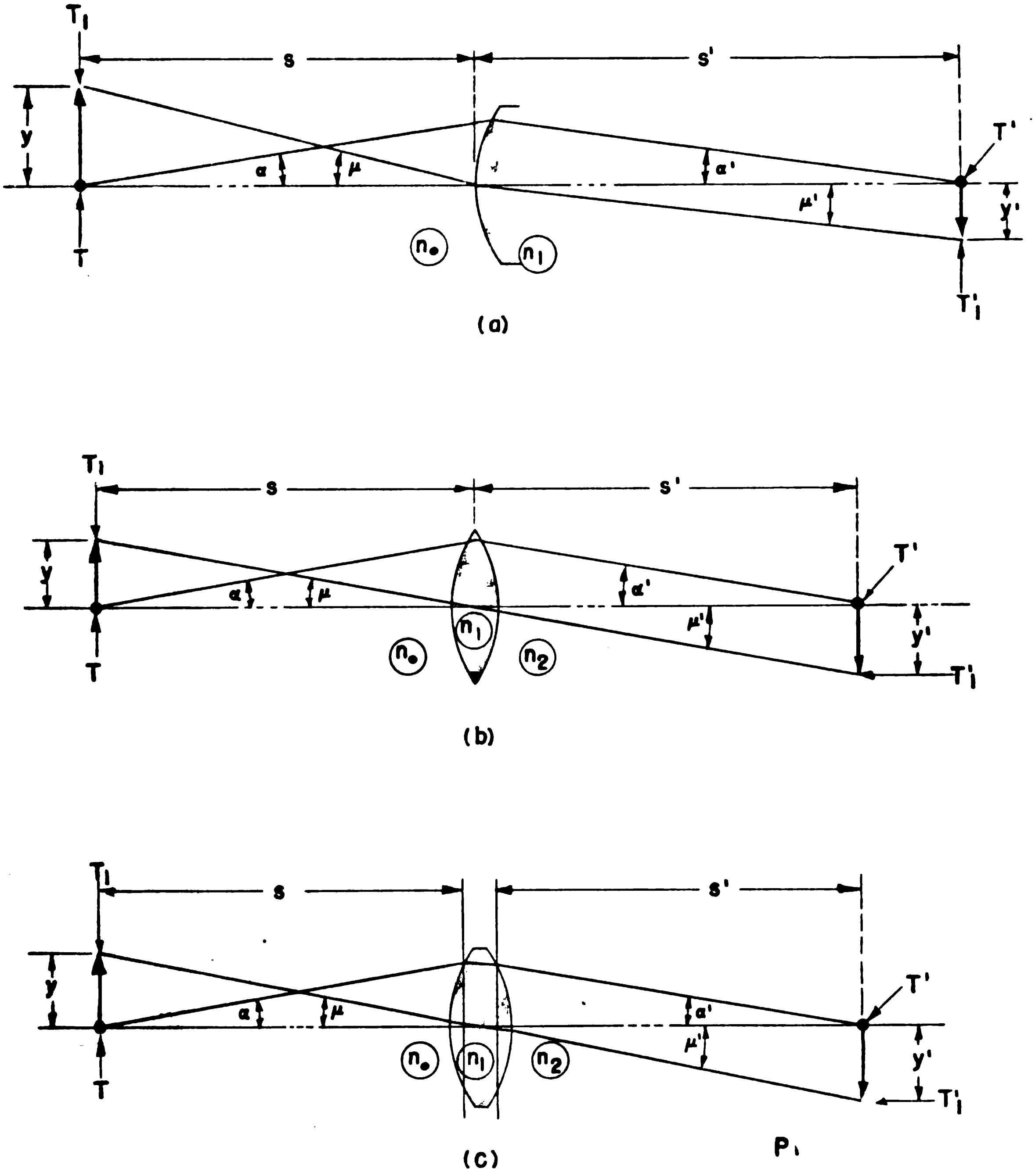


Figure 78. Angular ratio—lateral magnification.

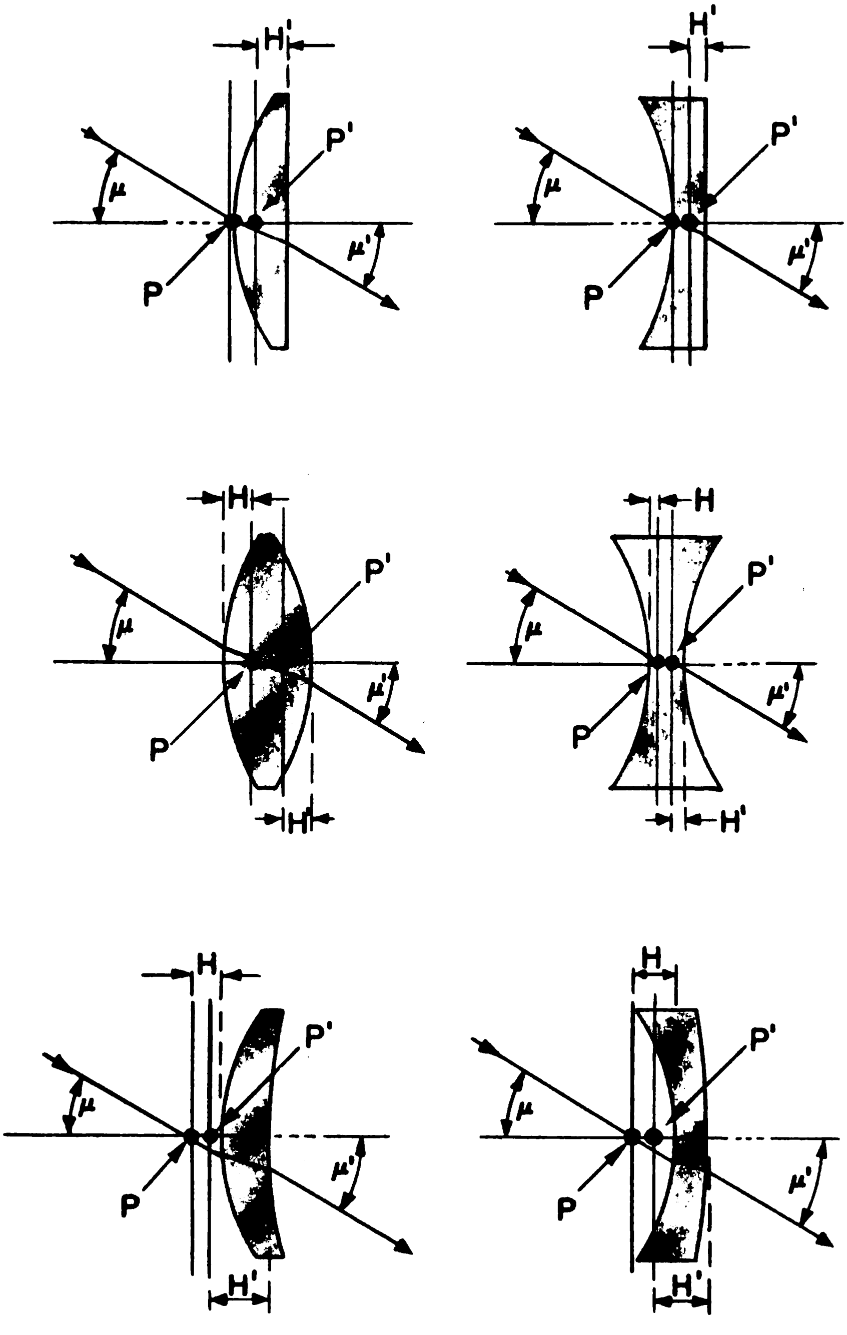


Figure 79. Gauss (principal) points—Gauss (principal) planes.

From these equations there has been derived the "*Helmholtz-Lagrange Law*" which states,

> "*The product of the refractive index, size of object, and inclination of the ray remain constant after any number of refractions.*"

This law is expressed by the following equation:

$$\alpha n_o y = \alpha' n_1 y' \tag{129}$$

56. Gauss (Principal) Points—Gauss (Principal) Planes

A pair of conjugate planes, normal to the optical axis, for which the lateral magnification $\beta = 1$, are called "*GAUSS PLANES*" or "*PRINCIPAL PLANES.*" In figure 79 these planes are indicated by the solid lines drawn through the points P and P′. The point P is known as the "*FIRST GAUSS POINT*" or "*FIRST PRINCIPAL POINT.*" The point P′ is known as the "*SECOND GAUSS POINT*" or "*SECOND PRINCIPAL POINT.*" In the case of a plane convex lens or a plane concave lens, one principal point lies at the vertex of the refracting surface, and the second point lies within the lens. This is illustrated in the upper two diagrams in figure 79. The principal planes lie within the lens if the lens is bi-convex or bi-concave as shown by the central diagrams. In the two lower diagrams are shown a positive and a negative meniscus lens; in both cases the principal planes lie outside the lens. The distance from the first surface to the first principal plane is designated by the letter H while H′ denotes the distance from the second principal point to the second refracting surface of the lens.

These principal points have the property that an incident ray in object space, directed towards the point P, will emerge from the lens parallel to the incident ray and its backward projection will intersect the axis at the point P′ assuming, however, that the lens is surrounded by one and the same medium, such as air. From this it follows that

$$u' = u \tag{130}$$

The principal points are of the greatest importance in the design of optical systems for various instruments since all distances between lenses are measured from these points. Formulae for computing the values of H and H′ will be given in Part 4, Thick Lens.

57. Nodal Points—Nodal Planes

The two conjugate points N and N′ (see fig. 80) were named by listing the "*NODAL POINTS.*" The planes drawn through these points, normal to the axis are known as the "*NODAL PLANES.*" These points and planes must be considered when the surface is bounded on one or both sides by a medium different from air. If both bounding media are the same, these points will have the same properties as the principal points, namely, a ray of light in object space directed towards the first nodal point N will emerge from the lens parallel to the incident ray. The emerging ray, projected backwards, will intersect the axis at the second nodal point N′. The distance from the front surface to the first nodal point is designated by the letter K, while K′ denotes the distance from the second nodal point to the rear surface.

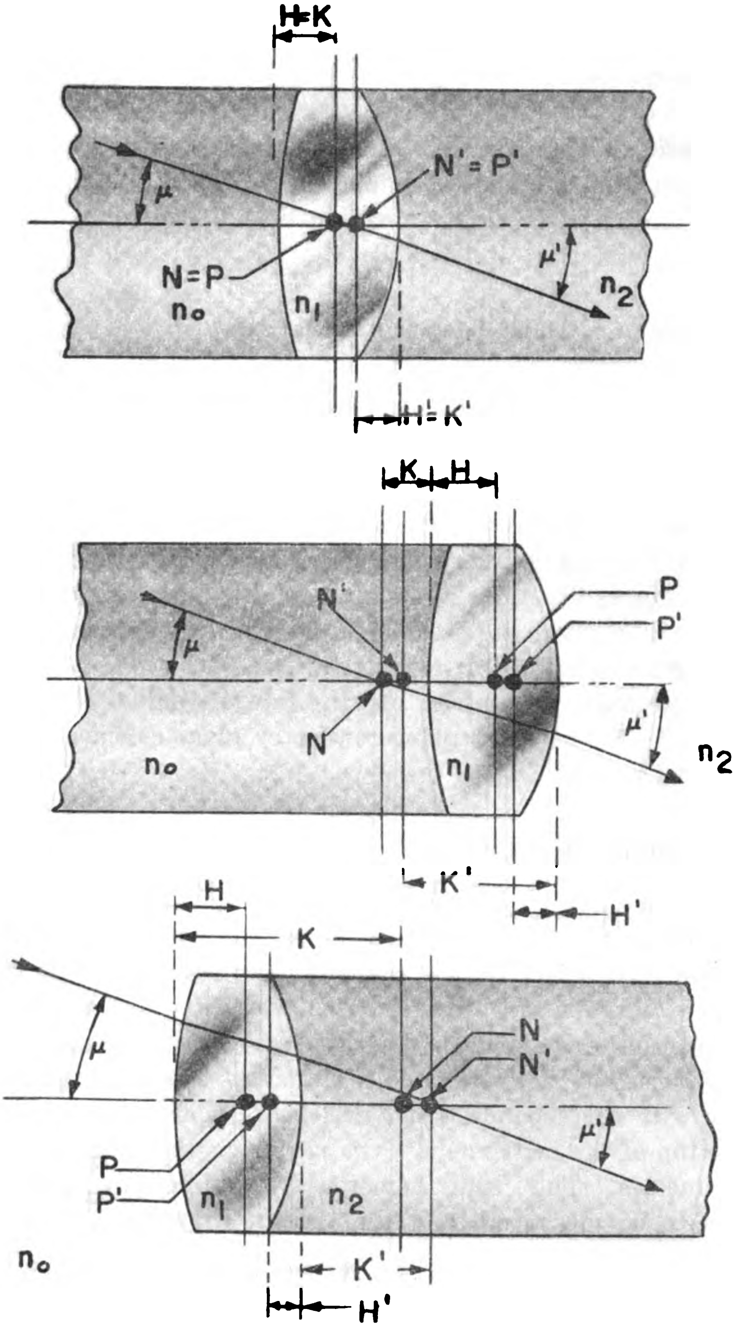


Figure 80. Nodal points—nodal planes.

When the first and last media are the same, then $n_o = n_2$ and the nodal points coincide with the principal points, a case illustrated in the upper diagram of figure 80. The first nodal point N is separated from the first focal point F′ by the distance f′, and f is the distance from the second nodal point N′ to the second focal point F. The nodal points will be situated to the left of the principal points if the first medium is denser than the last one. The case is illustrated in the central diagram of figure 80. If the last medium is denser than the first one the nodal points are located to the right of the principal points as shown in the lower diagram of figure 80. The computing of the values of K and K′ will be given in Part 4, Thick Lens.

58. Optical Center of a Thick Lens

Any ray of light emerging from a lens in a direction parallel to that of the incident ray will have passed through the *"OPTICAL CENTER"* of the lens. In figure 81 this point is designated by the letter M, and the distances from the first surface and the second surface to M are designated by the letters m_1 and m_2, respectively. Assuming the lens is surrounded by one and the same medium, then, in the case of the equi biconvex and equi bi-concave lenses, the optical center coincides with the geometrical center and $m_1 = m_2$, as illustrated in the upper diagrams of figure 81. In ordinary biconvex and bi-concave lenses the point M moves towards the vertex of the shorter radius (see central diagrams in fig. 81), and in the cases of positive and negative lenses the point M is situated inside the lens nearest the shorter radius. M lies on the vertex of the spherical surface in the cases of plano-convex or plano-concave lenses, as shown in the lower diagram of figure 81.

59. Measuring Radius of a Lens

The radius of a surface of a spherical lens is measured with an instrument known as the *"SPHEROMETER."* A well-fitting steel plunger (see fig. 82(a)) will slide in a vertical direction. Attached to this plunger is an engraved scale divided in tenths of a millimeter. The observer will read the scale with the aid of a small microscope with micrometer eye piece (not shown in the diagram), and readings may be taken to 0.001 millimeters. A series of rings of various diameters are usually supplied with the spherometer for use with corresponding size of lenses. The very accurately turned ring is situated at the top of the instrument. On each ring there are engraved the inside and the outside diameters. The inside diameter $2r_1$ is to be considered when the radius of a convex lens is to be measured, and the outside diameter $2r_2$ in the case of concave surfaces.

The surface to be tested is placed on the spherometer ring, while the spherical tip of the plunger is kept in contact with the surface by means of a counterweight suspended over small pulleys. Before measuring a spherical surface, place a plane plate on the selected ring and the plunger will move down. Adjust the instrument so that the zero line on the plunger scale will appear between the two parallel lines on the reticle of the microscope. Now place the convex (or concave) surface on the ring and read directly the height a (height of the segment) in the microscope. Since r is known, and a has been determined, we find the value of the radius as follows,

$$R = \frac{a^2 + r^2}{2a} \tag{131}$$

Other spherometers may vary in minor detail from the one described above.

Example 4:

The convex surface of a lens is to be measured with a ring whose inside diameter is 39.32 mm, hence, $r_1 = 19.66$ mm. With the aid of the microscope we find that the scale has been displaced in a downward direction to an amount of 5.225 mm, that is, $a = 5.225$ mm. Substituting these values of a and r_1 into equation (131) we have,

$$R = \frac{27.300625 + 386.5156}{10.45} = 39.5996 \text{ mm, or } R = 1.559 \text{ inches}$$

Example 5:

A concave surface is to be measured with the same ring whose outside diameter is 40.448 mm, hence, $r_2 = 20.224$ mm. The scale is displaced in an upward direction by an amount of 4.325 mm. With the aid of equation (131) we find,

$$R = \frac{18.705625 + 409.010176}{8.650} = 49.4669 \text{ mm, or } R = 1.947 \text{ inches}$$

PART 2. SPHERICAL SURFACE

60. Fundamental Formulae for Refraction by a Spherical Surface

$$\frac{n_1}{s'} - \frac{n_o}{s} = \frac{n_1}{f + x'} - \frac{n_o}{f' + x} = \frac{n_1 - n_o}{R} \tag{132}$$

$$n_1\sigma' - n_o\sigma = (n_1 - n_o)\,\rho \tag{133}$$

$$\frac{f'}{s} + \frac{f}{s'} = \frac{\sigma}{\phi'} + \frac{\sigma'}{\phi} = +1 \tag{134}$$

$$y = \frac{n_1 s y'}{n_o s'} = \frac{(s - R)\,y'}{s' - R} = -\frac{xy'}{f'} = -\frac{fy'}{x'} \tag{135}$$

$$y' = \frac{n_o s' y}{n_1 s} = \frac{(s' - R)\,y}{s - R} = -\frac{f'y}{x} = -\frac{x'y}{f} \tag{136}$$

$$\beta = \frac{y'}{y} = \frac{n_o s'}{n_1 s} = \frac{s' - R}{s - R} = -\frac{f'}{x} = -\frac{x'}{f} \tag{137}$$

$$\frac{s'}{s} = -\frac{f}{f' - s} = \frac{f}{x} = \frac{x'}{f'} \tag{138}$$

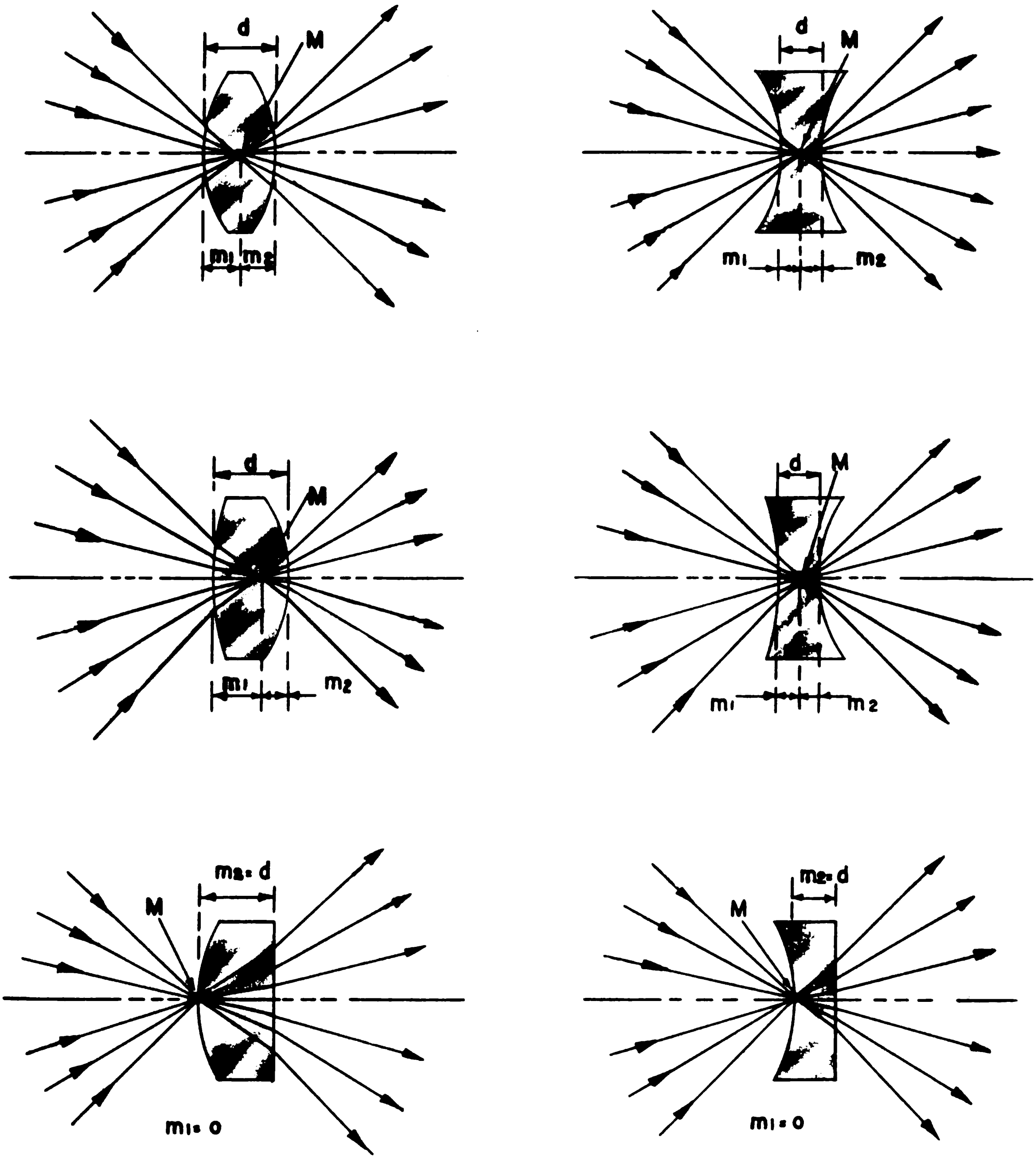


Figure 81. The optical center of a lens.

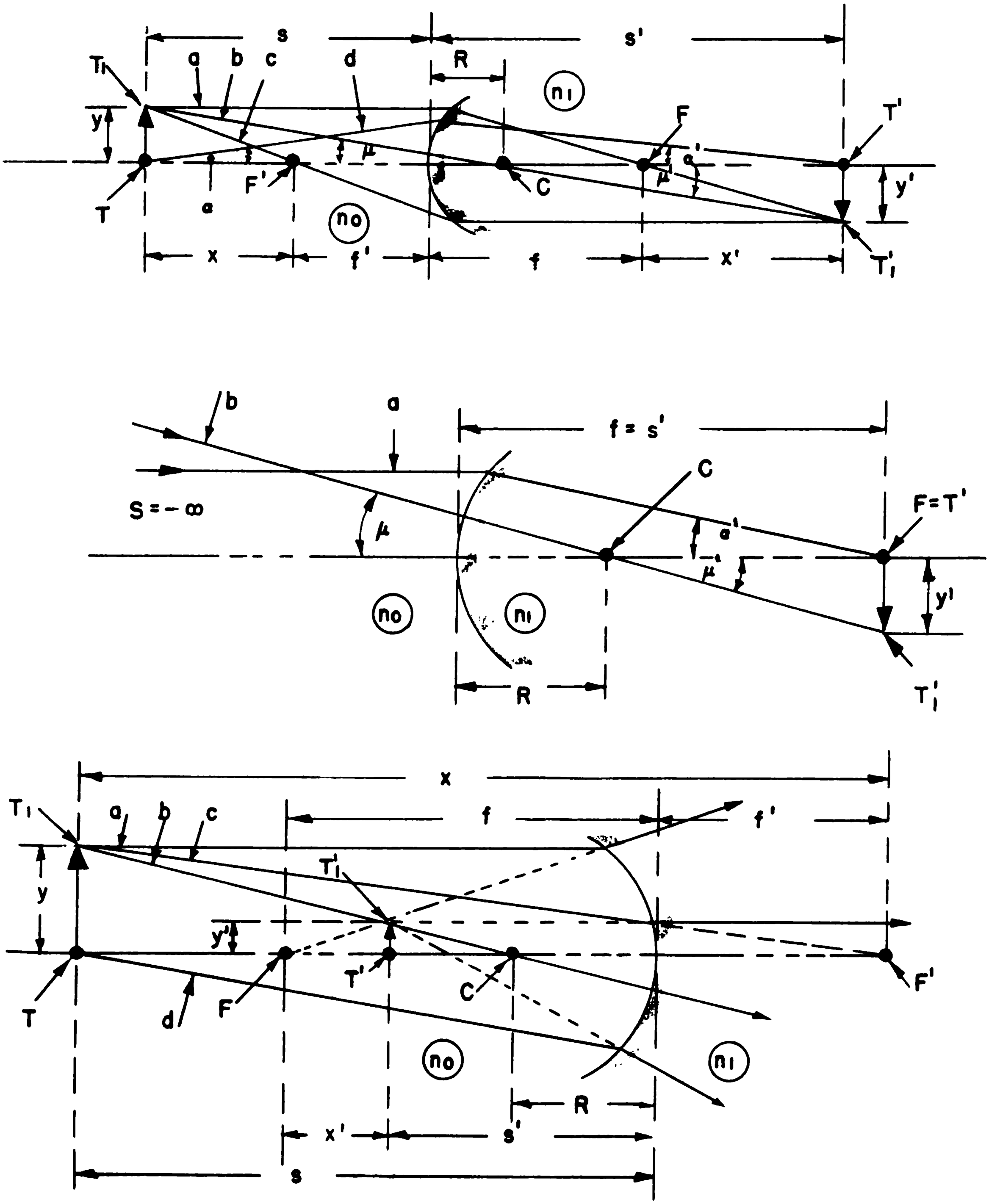


Figure 82. Readings on the spherometer.

$$xx' = ff' \tag{139}$$

$$f = \frac{n_1 R}{n_1 - n_o} \tag{140}$$

$$f' = -\frac{n_o R}{n_1 - n_o} \tag{141}$$

$$\phi = \frac{(n_1 - n_o)\rho}{n_1} = \frac{1}{f} \tag{142}$$

$$\phi' = -\frac{(n_1 - n_o)\rho}{n_o} = \frac{1}{f'} \tag{143}$$

$$\frac{f'}{f} = -\frac{n_o}{n_1} \tag{144}$$

$$\frac{\phi'}{\phi} = -\frac{n_1}{n_o} \tag{145}$$

$$s' = \frac{n_1 R s}{n_o R + (n_1 - n_o)\, s} = \frac{n_1 f s}{n_o f + n_1 s} \tag{146}$$

$$s = \frac{n_o R s'}{n_1 R - (n_1 - n_o)\, s'} = \frac{n_o f s'}{n_1 (f - s')} \tag{147}$$

$$\sigma' = \frac{n_o \sigma + (n_1 - n_o)\rho}{n_1} = \frac{n_1 \phi + n_o \sigma}{n_1} = \phi + \frac{n_o}{n_1}\sigma \tag{148}$$

$$\sigma = \frac{n_1 \sigma' - (n_1 - n_o)\rho}{n_o} = \frac{n_1 (\sigma' - \phi)}{n_o} \tag{149}$$

$$x = s - f' = -\frac{n_o f^2}{n_1 x'} = \frac{ff'}{x'} \tag{150}$$

$$x' = s' - f = -\frac{n_o f^2}{n_1 x} = \frac{ff'}{x} \tag{151}$$

Example 6:

A spherical convex surface, separating glass from air, has a radius of one inch. An object, extending 0.750 inches above the axis, is placed at a distance of 4.000 inches in front of the surface. Determine the values of f and f', s', x and x', and y'.

We have: $R = +1.000$ $s = -4.000$ $y = +0.750$ $n_o = 1$ (air)
$\rho = +1.000$ $\sigma = -0.250$ $n_1 = 1.5165$ (glass)

We apply the equations (140), (141), (146), (150), (151), and (136) and we find,

$f = +2.936$ $x = -2.064$
$f' = -1.936$ $s' = +5.690$ $x' = +2.754$ $y' = -0.703$

The (—) sign in front of the y' value indicates that the image has been inverted. These dimensions are shown in the upper diagram of figure 83.

Example 7:

The same spherical convex surface considered in the previous example will now receive light from a distant object. A principal ray coming from the tip of the target, including the angle μ with the axis will intersect the axis at the center c of the surface. What are the values of the various dimensions?

We have: $R = +1.000$ $n_0 = 1$ $\mu = -15°$

$s = \infty$ $n_1 = 1.5165$

$$f = s' = \frac{n_1 R}{n_1 - 1} = 2.936 \qquad f' = -\frac{R}{n_1 - 1} = -1.936$$

$$y' = +\ (f - R)\ \tan \mu = -0.519$$

The central diagram of figure 83 illustrates this example.

Example 8:

The spherical concave surface, shown in the lower diagram of figure 83, separates glass from air. The refractive index of the glass is equal to 1.6296 and the focal length of the surface is — 2.5883. The object has a height of 0.900. What are the object and image distances if the image shall have a height of 0.256 inches?

We have: $f = -2.588$ $y = +0.900$ $n_0 = 1$

$y' = +0.256$ $n_1 = 1.6296$

$$\text{From equation (140)}: \quad R = \frac{(n_1 - 1)\,f}{n_1} = -1.000 \text{ inch}$$

$$\text{From equation (141)}: \quad f' = -\frac{f}{n_1} = +1.588 \text{ inches}$$

Combining equations (134) and (137) we find

$$s = f' + \frac{fy}{n_1 y'} = -3.995 \text{ inches}$$

Substitute these values into equations (146), (150), and (151) and find

$$s' = \frac{n_1 f s}{f + n_1 s} = -1.852 \text{ inches}$$

$$x = s - f' = -5.583 \text{ inches}$$

$$x' = s' - f = +0.736 \text{ inches}$$

61. Construction of Image

With the above constants computed we can easily construct the image for any object distance.

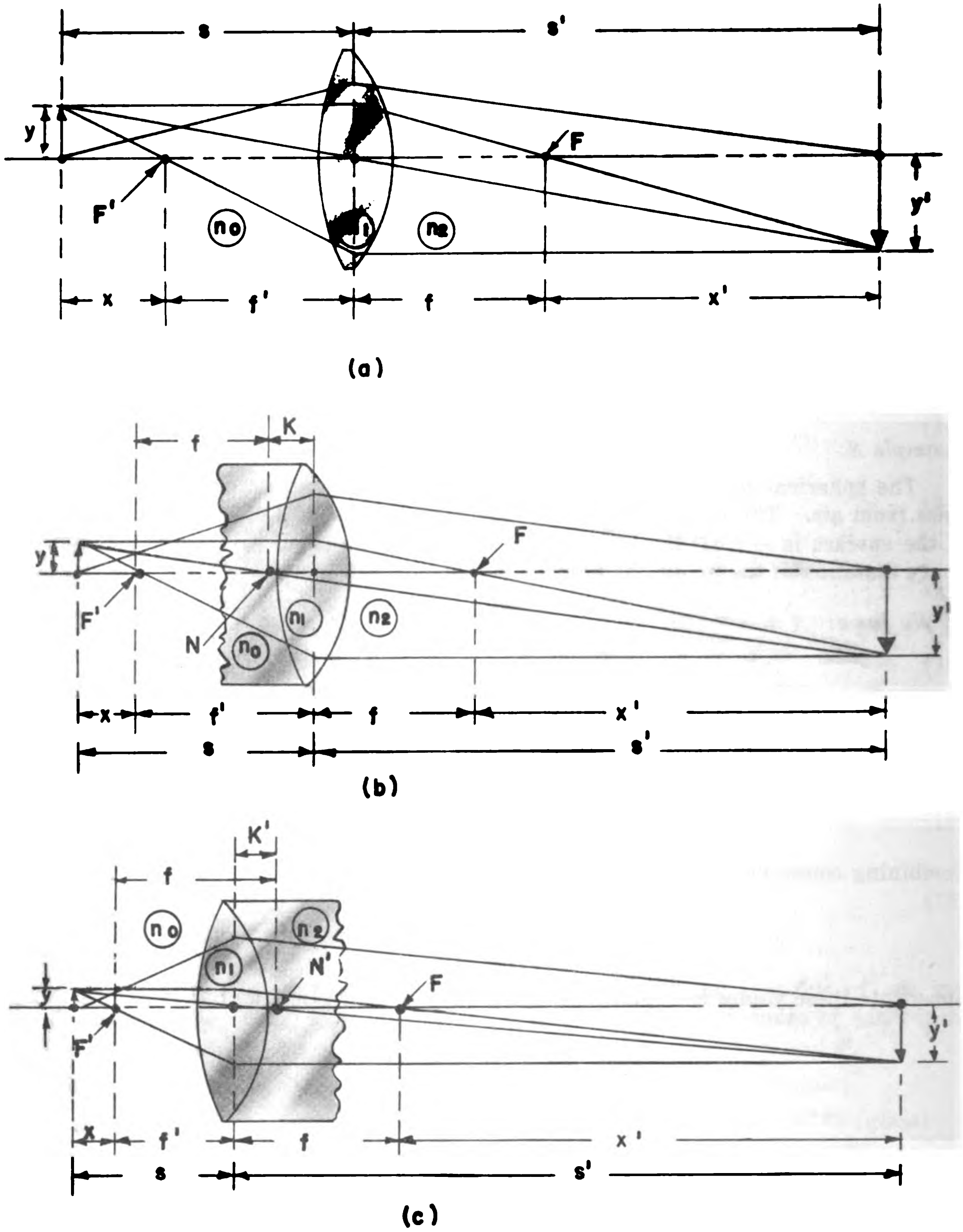


Figure 83. Refractions at a spherical surface.

The object is located at the distance s in front of a spherical convex surface. See example 6 and the upper diagram in figure 83. The rays a, b, and c proceed from the target point T_1. Draw the ray a parallel to the optical axis to the refracting surface. After a refraction at this surface it will proceed in the second medium, pass through the second focal point F, and then pierce the image plane at the point T_1' at the distance y′ below the axis. T_1' is the conjugate image point to the object point T_1. The second ray b, also proceeding from the object point T_1, is directed towards the center of curvature c. This ray is normal to the surface and, therefore, the ray will not be refracted but will follow the same direction in the second medium, and will finally pierce the image plane at the same point T_1'. The ray c, also proceeding from T_1, will pass through the first focal point F′ and then, after a refraction at the surface, it will proceed in the second medium parallel to the optical axis. It will pierce the image plane also at the point T_1'. The ray marked d, proceeding from the axial object point T will be refracted at the surface, proceed in the second medium, and pierce the image plane at the axial point T′. $T' - T_1'$ is the image of the target $T = T_1$. Hence, we need consider only two rays in order to construct the image.

The object is located at an infinite distance, $s = -\infty$, see example 7 and the center diagram of figure 83. To construct the image we draw the ray a parallel to the axis. After a refraction at the surface it will proceed in the second medium and intersect the axis at the second focal point F, which is also the image point T′ of the axial point T. The ray b, proceeding from the object point T_1, travels in the direction of the center C. It will pass through the surface undeviated and, proceeding in the second medium, it will finally pierce the image plane at the point T_1'. Hence, $T' - T_1'$ is the image of the target.

If the object is located at a finite distance from a spherical concave refracting surface (see example 8 and the lower diagram of figure 83) the method for constructing the image is identical with the one for the convex surface, except that in this case the various points are determined by the emerging rays projected backwards. The image $T' - T_1'$ is called a *"VIRTUAL IMAGE."*

PART 3. THIN LENS

62. Fundamental Formulae for Refraction by a Thin Lens

The combination of two refracting surfaces bounding some optical medium forms a lens. If the thickness of the lens is small or assumed equal to zero the following equations may be used:

$$s = f' + x = \frac{s'x}{f} = \frac{n_o f s'}{n_2 (f - s')} \tag{152}$$

$$s' = f + x' = \frac{sf}{x} = \frac{n_2 f s}{n_o f + n_2 s} \tag{153}$$

$$\sigma = \frac{n_2 (\sigma' - \phi)}{n_o} \tag{154}$$

$$\sigma' = \phi + \frac{n_o \sigma}{n_2} \tag{155}$$

$$x = s - f' = \frac{ff'}{x'} = -\frac{n_o f^2}{n_2 x'} \tag{156}$$

$$x' = s' - f = \frac{ff'}{x} = -\frac{n_o f^2}{n_2 x} \tag{157}$$

$$y = -\frac{n_2 s y'}{n_o s'} \tag{158}$$

$$y' = -\frac{n_o s' y}{n_2 s} \tag{159}$$

$$\beta = \frac{y'}{y} = -\frac{n_o s'}{n_2 s} \tag{160}$$

The following five equations express mathematically, in various forms, the *"LAW OF THE CONJUGATE FOCI."*

$$\frac{n_o}{s} + \frac{n_2}{f} = \frac{n_2}{s'} \tag{161}$$

$$n_o \sigma + n_2 \phi = n_2 \sigma' \tag{162}$$

$$\frac{n_o n_2}{n_2 x - n_o f} + \frac{n_2}{f} = \frac{n_2}{f + x'} \tag{163}$$

$$f = \frac{n_2 s s'}{n_2 s - n_o s'} \tag{164}$$

$$\phi = \sigma' - \frac{n_o \sigma}{n_2} \tag{165}$$

If the lens is surrounded by air, as is usually the case in fire-control instruments, then $n_o = n_2 = 1$, and the formulae given above may be written as:

$$\frac{1}{s} + \frac{1}{f} = \frac{1}{s'} \tag{166}$$

$$\text{or } \sigma + \phi = \sigma' \tag{167}$$

$$f = -f' = \frac{ss'}{s - s'} \tag{168}$$

$$s = \frac{fs'}{f - s'} \qquad \text{or,} \tag{169}$$

$$\frac{1}{x - f} + \frac{1}{f} = \frac{1}{f + x'} \tag{170}$$

$$s' = \frac{fs}{f + s} \tag{171}$$

$$f^2 = - xx' \tag{172}$$

$$\beta = \frac{y'}{y} = - \frac{s'}{s} \tag{173}$$

63. Formulae for Computing Constants of a Thin Lens in Terms of Refractive Indices and Radii R, R_1 and R_2

The following general equations refer to cases where the three indices differ from each other:

$$f = + \frac{n_2 R_1 R_2}{(n_2 - n_1) R_1 + (n_1 - n_o) R_2} = - \frac{n_2 f'}{n_o} \tag{174}$$

$$\phi = + \frac{(n_1 - n_o) \rho_1 + (n_2 - n_1) \rho_2}{n_2} = - \frac{n_o \phi'}{n_2} \tag{175}$$

$$f' = - \frac{n_o R_1 R_2}{(n_2 - n_1) R_1 + (n_1 - n_o) R_2} = - \frac{n_o f}{n_2} \tag{176}$$

$$\phi' = - \frac{(n_1 - n_o) \rho_1 + (n_2 - n_1) \rho_2}{n_o} = - \frac{n_2 \phi}{n_o} \tag{177}$$

When f and R_1 are known

$$R_2 = \frac{(n_2 - n_1) f R_1}{n_2 R_1 - (n_1 - n_o) f} \tag{178}$$

$$\rho_2 = \frac{n_2 \phi - (n_1 - n_o) \rho_1}{n_2 - n_1} \tag{179}$$

When f and R_2 are known

$$R_1 = \frac{(n_1 - n_o) f R_2}{n_2 R_2 - (n_2 - n_1) f} \tag{180}$$

$$\rho_1 = \frac{n_2 \phi - (n_2 - n_1) \rho_2}{n_1 - n_o} \tag{181}$$

Case 1: $R_2 = - R_1$

that is, the lens is either equi bi-convex or equi bi-concave

$$f = + \frac{n_2 R_1}{(n_1 - n_o) - (n_2 - n_1)} \tag{182}$$

$$\phi = + \frac{[(n_1 - n_o) - (n_2 - n_1)] \rho_1}{n_2} \tag{183}$$

$$f' = - \frac{n_o R_1}{(n_1 - n_o) - (n_2 - n_1)} \tag{184}$$

$$\phi' = - \frac{[(n_1 - n_o) - (n_2 - n_1)] \rho_1}{n_o} \tag{185}$$

When f and n are known

$$R_1 = -R_2 = \frac{[(n_1 - n_o) - (n_2 - n_1)]\,f}{n_2} \tag{186}$$

$$\rho_1 = -\rho_2 = \frac{n_2\phi}{(n_1 - n_o) - (n_2 - n_1)} \tag{187}$$

Case 2: $R_2 = rR_1$

that is, the radii have a definite ratio to each other. If

r = − 1, then the lens is equi bi-convex or equi bi-concave

r = + 1, then the lens is of zero power

r = negative, then the lens is bi-convex or bi-concave

r = positive, then the lens is either positive or negative meniscus

$$f = \frac{n_2 r R_1}{(n_1 - n_o) r + (n_2 - n_1)} \tag{188}$$

$$\phi = \frac{[(n_1 - n_o) r + (n_2 - n_1)]\,\rho_1}{n_2 r} \tag{189}$$

$$f' = -\frac{n_o r R_1}{(n_1 - n_o) r + (n_2 - n_1)} \tag{190}$$

$$\phi' = -\frac{[(n_1 - n_o) r + (n_2 - n_1)]\,\rho_1}{n_o r} \tag{191}$$

When f and n are known

$$R_1 = \frac{[(n_1 - n_o)\,r + (n_2 - n_1)]\,f}{n_2 r} \tag{192}$$

$$\rho_1 = \frac{n_2 r \phi}{(n_1 - n_o)\,r + (n_2 - n_1)} \tag{193}$$

Case 3: $R_1 = \pm\,\infty$

that is, the first surface is plane

$$f = +\frac{n_2 R_2}{n_2 - n_1} = -\frac{n_2 f'}{n_o} \tag{194}$$

$$\phi = +\frac{(n_2 - n_1)\,\rho_2}{n_2} = -\frac{n_o \phi'}{n_2} \tag{195}$$

$$f' = -\frac{n_o R_2}{n_2 - n_1} = -\frac{n_o f}{n_2} \tag{196}$$

$$\phi' = -\frac{(n_2 - n_1)\,\rho_2}{n_o} = -\frac{n_2 \phi}{n_o} \tag{197}$$

When f and n are known

$$R_2 = \frac{(n_2 - n_1)\, f}{n_2} \tag{198}$$

$$\rho_2 = \frac{n_2 \phi}{n_2 - n_1} \tag{199}$$

Case 4: $R_2 = \pm \infty$

that is, the second surface is plane

$$f = + \frac{n_2 R_1}{n_1 - n_o} = - \frac{n_2 f'}{n_o} \tag{200}$$

$$\phi = + \frac{(n_1 - n_o)\, \rho_1}{n_2} = - \frac{n_o \phi'}{n_2} \tag{201}$$

$$f' = - \frac{n_o R_1}{n_1 - n_o} = - \frac{n_o f}{n_2} \tag{202}$$

$$\phi' = - \frac{(n_1 - n_o)\, \rho_1}{n_o} = - \frac{n_2 \phi}{n_o} \tag{203}$$

When f and n are given

$$R_1 = \frac{(n_1 - n_o)\, f}{n_2} \tag{204}$$

$$\rho_1 = \frac{n_2 \phi}{n_1 - n_o} \tag{205}$$

When the thin lens is surrounded by air, $n_o = n_2 = 1$ and the general equations reduce to the following:

$$f = \frac{R_1 R_2}{(n_1 - 1)(R_2 - R_1)} = - f' \tag{206}$$

$$\phi = + (n_1 - 1)(\rho_1 - \rho_2) = - \phi' \tag{207}$$

$$f' = + \frac{R_1 R_2}{(n_1 - 1)(R_1 - R_2)} = - f \tag{208}$$

$$\phi' = + (n_1 - 1)(\rho_2 - \rho_1) = - \phi \tag{209}$$

When f and R_1 are known

$$R_2 = + \frac{(n_1 - 1)\, f R_1}{(n_1 - 1)\, f - R_1} \tag{210}$$

$$\rho_2 = \rho_1 - \frac{\phi}{n_1 - 1} \tag{211}$$

When f and R_2 are known

$$R_1 = \frac{(n_1 - 1)\, f R_2}{R_2 + (n_1 - 1)\, f} \tag{212}$$

$$\rho_1 = \rho_2 + \frac{\phi}{n_1 - 1} \tag{213}$$

Case 1: $R_2 = - R_1$

that is, the lens is either equi bi-convex or equi bi-concave

$$f = + \frac{R_1}{2(n_1 - 1)} \tag{214}$$

$$\phi = + 2(n_1 - 1)\rho_1 \tag{215}$$

$$f' = - \frac{R_1}{2(n_1 - 1)} \tag{216}$$

$$\phi' = - 2(n_1 - 1)\ \rho_1 \tag{217}$$

When f and n are known

$$R_1 = - R_2 = 2(n_1 - 1)\ f \tag{218}$$

$$\rho_1 = - \rho_2 = \frac{\phi}{2(n_1 - 1)} \tag{219}$$

Case 2: $R_2 = rR_1$

that is, the radii have a definite ratio to each other. If

$r = - 1$, then lens is equi bi-convex or equi- bi-concave

$r = + 1$, then the lens is of zero power

$r =$ negative, then lens is bi-convex or bi-concave

$r =$ positive, then lens is either positive or negative meniscus

$$f = \frac{rR_1}{(n_1 - 1)(r - 1)} \tag{220}$$

$$\phi = \frac{(n_1 - 1)(r - 1)\ \rho_1}{r} \tag{221}$$

$$f' = - \frac{rR_1}{(n_1 - 1)(r - 1)} \tag{222}$$

$$\phi' = - \frac{(n_1 - 1)(r - 1)\ \rho_1}{r} \tag{223}$$

When f and n are known

$$R_1 = \frac{(n_1 - 1)(r - 1)\ f}{r} \tag{224}$$

$$\rho_1 = \frac{r\phi}{(n_1 - 1)(r - 1)} \tag{225}$$

Case 3: $R_1 = \pm \infty$

that is, the first surface is plane

$$f = -\frac{R_2}{n_1 - 1} \tag{226}$$

$$\phi = -(n_1 - 1)\,\rho_2 \tag{227}$$

$$f' = +\frac{R_2}{n_1 - 1} \tag{228}$$

$$\phi' = +(n_1 - 1)\,\rho_2 \tag{229}$$

When f and n are known

$$R_2 = -(n_1 - 1)\,f \tag{230}$$

$$\rho_2 = -\frac{\phi}{n_1 - 1} \tag{231}$$

Case 4: $R_2 = \pm \infty$

that is, the second surface is plane

$$f = +\frac{R_1}{n_1 - 1} \tag{232}$$

$$\phi = +(n_1 - 1)\,\rho_1 \tag{233}$$

$$f' = -\frac{R_1}{n_1 - 1} \tag{234}$$

$$\phi' = -(n_1 - 1)\,\rho_1 \tag{235}$$

When f and n are known

$$R_1 = (n_1 - 1)\,f \tag{236}$$

$$\rho_1 = \frac{\phi}{n_1 - 1} \tag{237}$$

In the following four examples a thin lens of radii $R_1 = 2.5$ inches and $R_2 = -1.5$ inches and refractive index $n_1 = 1.5165$ is placed in various media. Locate the image and find its size if $x = -1.0$ inch and $y = 0.5$ inch. Equations (152), (153), (157), (159), (160), (174), (176), (206), and (207) will be used.

Example 9: $n_o = n_2 = 1$ (air)

$$f = -f' = \frac{R_1R_2}{(n_1 - 1)\,(R_2 - R_1} = +1.815 \qquad \phi = -\phi' = (n_1 - 1)\,(\rho_1 - \rho_2) = +0.551$$

$$x' = \frac{ff'}{x} = +3.294$$

$$s = f' + x = -2.815 \qquad s' = f + x' = +5.109$$

$$y' = \frac{s'y}{s} = -0.907 \qquad K = f' + f = 0$$

See figure 84(a).

Example 10: $n_o = 1.3336$ $n_2 = 1$

$$f = \frac{R_1R_2}{(n_1 - n_o)\,R_2 - (n_1 - 1)\,R_1} = +2.395 \qquad f' = -n_of = -3.194$$

$$x' = \frac{ff'}{x} = +7.650$$

$$s = f' + x = -4.194 \qquad s' = f + x' = +10.045$$

$$y' = \frac{n_1s'y}{s} = -1.597 \qquad K = f' + f = -0.799$$

See figure 84(b).

Example 11: $n_o = 1$ $n_2 = 1.333$

$$f = \frac{n_2R_1R_2}{(n_2 - n_1)\,R_1 + (n_1 - 1)\,R_2} = +4.059 \qquad f' = -\frac{f}{n_2} = -3.044$$

$$x' = \frac{ff'}{x} = +12.356$$

$$s = f' + x = -4.044 \qquad s' = f + x' = +16.415$$

$$y' = \frac{s'y}{n_2s} = -1.522 \qquad K' = f' + f = +1.015$$

See figure 84(c).

Example 12: $n_o = n_2 = 1.3336$

$$f = -f' = \frac{n_2R_1R_2}{(n_1 - n_o)\,(R_2 - R_1)} = +6.836$$

$$x' = -\frac{f^2}{x} = +46.731$$

$$s = f' + x = -7.836 \qquad s' = f + x' = +53.567$$

$$y' = \frac{s'y}{s} = -3.418 \qquad K = K' = 0$$

Example 13:

The equi bi-concave lens shown in figure 85(a) is in air and its focal length is to be equal to -1.5 inches. What are the radii of the lens when the refractive index of the glass is $n_1 = 1.6170$?

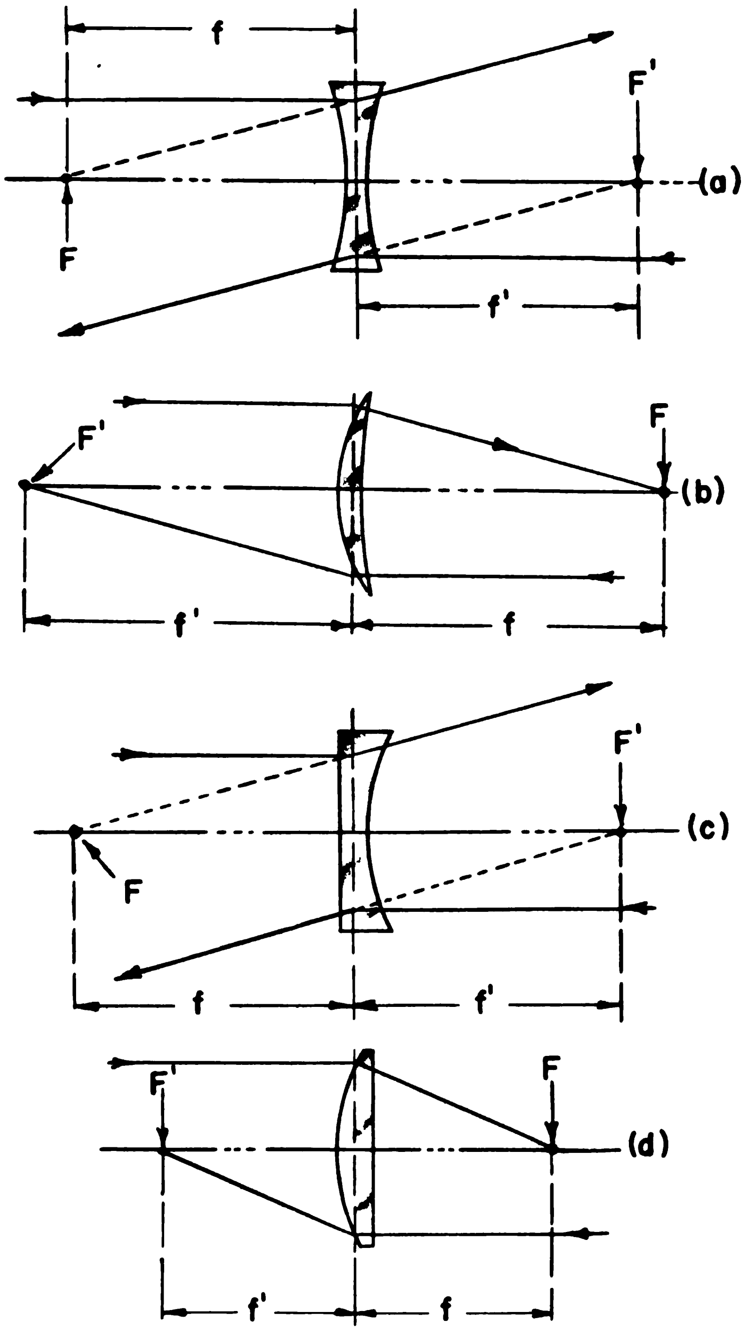


Figure 84. Refractions by a thin lens.

We have: $f = -f' = -1.500$ $n_1 = 1.6170$

Substitute these values into equation (218) and find

$$R_1 = -R_2 = 2(n_1 - 1)\,f = -1.851$$

Example 14:

The second radius of the thin collective lens shown in figure 85(b) is to be six times as long as the first radius. The focal length of the lens, which is surrounded by air, is to be 2 inches. What are the radii of the lens when the refractive index of the glass is 1.5170?

We have $R_2 = rR_1$ $f = -f' = 2.000$

$r = 6$ $n_1 = 1.5170$

Applying equation (224) we find

$$R_1 = \frac{(n_1 - 1)(r - 1)\,f}{r} = +0.862 \qquad R_2 = rR_1 = +5.172$$

Example 15:

The plane-concave lens illustrated in figure 85(c) is in air and the second surface has a radius of one inch. What are the focal lengths of the lens when the refractive index of the glass is equal to 1.6170?

We have $R_1 = \pm\infty$, $R_2 = +1$ inch and $n_1 = 1.6170$

With the aid of equation (226) we find,

$$f = -f' = -\frac{R_2}{n_1 - 1} = -1.621$$

Example 16:

In figure 85(d) is shown a convex-plane lens in air. The focal length of the lens is one inch and the refractive index is equal to 1.5170. What is the radius of the first surface?

We have $f = -f' = +1$ inch $R_2 = \pm\infty$ $n_1 = 1.5170$

Apply equation (236) and find,

$$R_1 = (n_1 - 1)\,f = 0.5170$$

64. Construction of Image

The method of constructing the image formed by a thin lens is identical with that applied in the case of a single spherical reflecting surface (see figure 83) except that the undeviated rays pass through the geometrical center of the lens. This will readily be understood from the diagrams in figure 84. In the cases when $n_0 \lessgtr n_2$ the principal ray will not pass through the center of the lens but through the nodal points N or N′, respectively (see fig. 80(b) and (c).

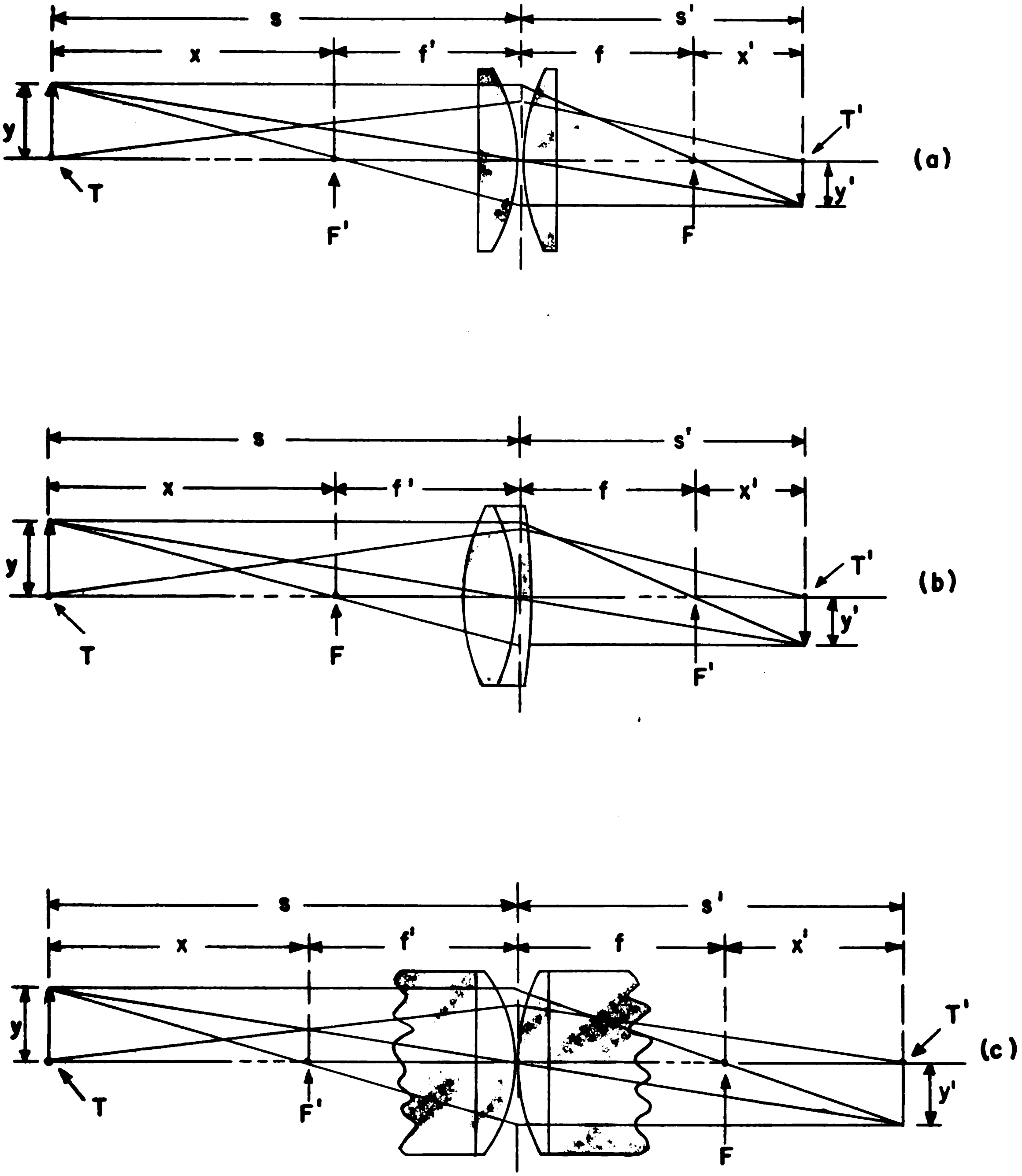


Figure 85. Refractions by a thin lens.

65. Two Thin Lenses in Contact

The diagrams in figure 86 illustrate three lens systems, each consisting of two thin lenses in contact. In the discussions following, it is assumed that the focal lengths f_1 and f_2 of the front and rear lens, as well as the object distance s and the height y of the object are known. The focal lengths and powers of the system are computed as follows:

$$f = \frac{f_1 f_2}{f_1 + f_2} = - f' \tag{238}$$

$$\phi = - \phi' = \phi_1 + \phi_2 \tag{239}$$

If the lenses are surrounded by water so that $n_o = n_3 = 1.333$ as illustrated in figure 86(c), then the above formulae are written as follows:

$$f = \frac{n_3 f_1 f_2}{n_3 f_1 + f_2} \tag{240}$$

$$\phi = \frac{\phi_1}{n_3} + \phi_2 \tag{241}$$

$$f' = - \frac{n_o f_1 f_2}{n_3 f_1 + f_2} \tag{242}$$

$$\phi' = - \frac{\phi_1 + n_3 \phi_2}{n_o} \tag{243}$$

Example 17:

The focal lengths in air of the two thin plane-convex lenses shown in figure 86(a) are 2 inches and 3 inches, respectively. The object, 0.5 inches high, is located at a distance of 3.2 inches from the point of contact. What are the constants of the lens system?

We have $f_1 = 2$, $f_2 = 3$, $s = - 3.2$ and $y = 0.5$

Substituting these values into equations (238), (156), (153), (157), and (159) we find

$$f = - f' = \frac{f_1 f_2}{f_1 + f_2} = 1.20 \qquad x = s - f' = - 2.00$$

$$s' = \frac{sf}{x} = + 1.92$$

$$x' = s' - f = + 0.72 \qquad y' = \frac{s'y}{s} = - 0.30$$

Example 18:

The same two lenses considered in the preceding paragraph are surrounded by water, $n_o = n_3 = 1.333$, as illustrated in figure 86(c). How will this change affect the constants of the lens? Using equation (240) and proceeding as before we find:

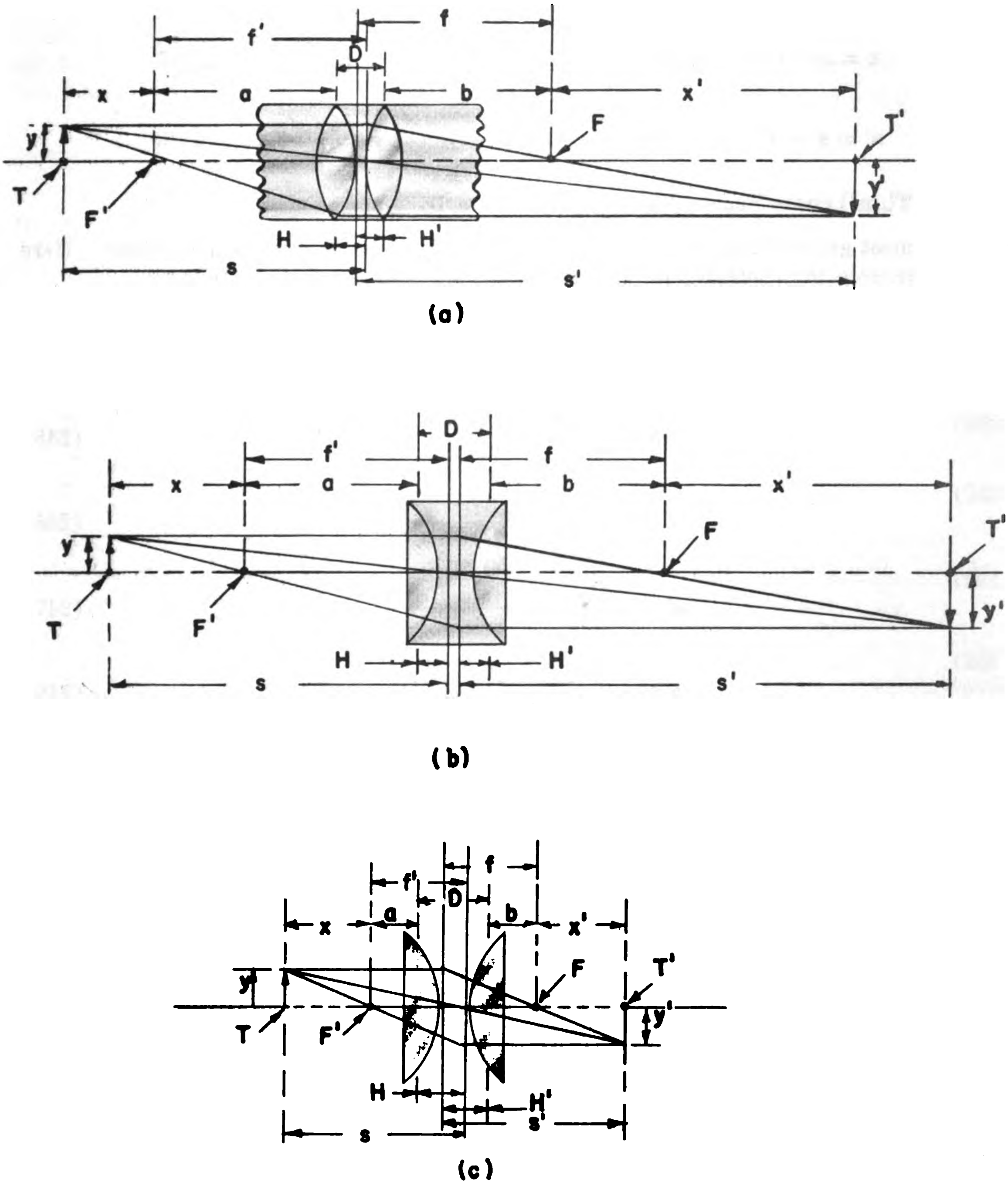


Figure 86. Two thin lenses in contact.

$$f = -f' = \frac{n_3 f_1 f_2}{n_3 f_1 + f_2} = 1.412$$

$$x = s - f' = -1.788 \qquad s' = \frac{fs}{x} = +2.527$$

$$x' = s' - f = +1.115 \qquad y' = \frac{n_o s' y}{n_8 s} = -0.395$$

66. Two Thin Lenses Separated by the Distance D

The most general case is when the media surrounding the lenses are unlike. Here we can evaluate the constants of the system by the equations which follow.

$$f = \frac{n_4 f_1 f_2}{n_4 (f_1 - D) + n_2 f_2} \tag{244}$$

$$\phi = \frac{n_4 \phi_2 (1 - D\phi_1) + n_2 \phi_1}{n_4} \tag{245}$$

$$f' = -\frac{n_o f_1 f_2}{n_4 (f_1 - D) + n_2 f_2} \tag{246}$$

$$\phi' = -\frac{n_4 \phi_2 (1 - D\phi_1) + n_2 \phi_1}{n_o} \tag{247}$$

$$a = -\frac{n_o (n_2 f_2 - n_4 D)\, f_1}{n_2 [n_2 f_2 + n_4 (f_1 - D)]} \tag{248}$$

$$b = \frac{n_4 f_2 (f_1 - D)}{n_4 (f_1 - D) + n_2 f_2} \tag{249}$$

$$H = a - f' = \frac{n_o n_4 f_1 D}{n_2 [n_2 f_2 + n_4 (f_1 - D)]} \tag{250}$$

$$H' = b - f = -\frac{n_4 f_2 D}{n_4 (f_1 - D) + n_2 f_2} \tag{251}$$

$$x = \frac{n_o f f'}{n_4 x'} \tag{252}$$

$$x' = \frac{n_o f f'}{n_4 x} \tag{253}$$

$$s = f' + x \tag{254}$$

$$s' = f + x' \tag{255}$$

$$y' = \frac{s' y}{s} \tag{256}$$

If the lens system is in air then $n_o = n_2 = n_4 = 1$, and the above formulae may be written as follows:

$$f_1 = -f_1' = \frac{R_1R_2}{(n_1 - 1)(R_2 - R_1)} \qquad (257)$$

$$\phi_1 = -\phi_1' = (n_1 - 1)(\rho_1 - \rho_2) \qquad (258)$$

$$f_2 = -f_2' = \frac{R_3R_4}{(n_3 - 1)(R_4 - R_3)} \qquad (259)$$

$$\phi_2 = -\phi_2' = (n_3 - 1)(\rho_3 - \rho_4) \qquad (260)$$

$$f = -f' = \frac{f_1f_2}{f_1 + f_2 - D} \qquad (261)$$

$$\phi = -\phi' = \phi_1 + \phi_2 - D\phi_1\phi_2 \qquad (262)$$

$$a = -\frac{(f_2 - D)f_1}{f_1 + f_2 - D} \qquad (263)$$

$$b = \frac{(f_1 - D)f_2}{f_1 + f_2 - D} \qquad (264)$$

$$H = \frac{Df_1}{f_1 + f_2 - D} \qquad (265)$$

$$H' = -\frac{Df_2}{f_1 + f_2 - D} \qquad (266)$$

$$x = -\frac{f^2}{x'} \qquad (267)$$

$$x' = -\frac{f^2}{x} \qquad (268)$$

When D, f and f_1 are known, then $f_2 = \frac{(f_1 - D)f}{f_1 - f}$ (269)

When D, f and f_2 are known, then $f_1 = \frac{(f_2 - D)f}{f_2 - f}$ (270)

When f, f_1 and f_2 are known, then $D = f_1 + f_2 - \frac{f_1f_2}{f}$ (271)

D and f are given, and $f_1 = f_2$, then $f_1 = f_2 = f + \sqrt{f(f - D)}$ (272)

Given f and $f_1 = f_2$, then $D = \frac{(2f - f_1)f_1}{f}$ (273)

Given D and f, and $f_2 = rf_1$, $f_1 = \dfrac{f(r+1) \pm \sqrt{f[(r+1)^2 f - 4rD]}}{2r}$ (274)

When f and f_1 are given, and $f_2 = rf_1$, then $D = \dfrac{[(r+1)f - rf_1]f_1}{f}$ (275)

In figure 88(a) is shown a thin lens system consisting of a collective and a dispersive lens, separated by the distance D. Combinations of this type are used in the so-called *"TELE-OBJECTIVE."* This system has one great advantage over the ordinary two-lens objective, namely, the distance L from the collective lens to the image plane is much shorter than the focal length of the combination. If the system is in air, as is usually the case, we determine the various constants with the aid of the following formulae:

$$f_1 = \frac{(f_2 - D)f}{f_2 - f} \tag{276}$$

$$f_2 = \frac{bD}{f - L} = \frac{(D - L)D}{f - L} \tag{277}$$

$$y' = - f \tan \mu \tag{278}$$

Example 19:

The two bi-convex lenses shown in figure 87(a) are separated by the distance D and are surrounded by water. The known constants are:

$R_1 = + 2.500$ $R_3 = + 2.250$ $D = 1.000$ $n_o = n_2 = n_4 = 1.333$

$R_2 = - 2.000$ $R_4 = - 2.750$ $y = 0.750$ $n_1 = n_3 = 1.5170$

Applying equations (174), (175), (176), and (177), and then equations (244) to (256) we find:

$f_1 = - f_1' = + 8.079$ $\phi_1 = - \phi_1' = + 0.124$ $f = - f' = + 4.522$

$f_2 = - f_2' = + 8.999$ $\phi_2 = - \phi_2' = + 0.111$ $\phi = - \phi' = + 0.221$

$a = - 4.019$ $H = + 0.503$ $x = - 2.000$ (assumed) $s = - 6.522$

$b = + 3.962$ $H' = - 0.560$ $x' = + 10.224$ $s' = + 14.746$

$y' = - 1.696$

Example 20:

The two plane-convex lenses shown in figure 87(b) are separated by the distance D. The medium between the lenses is water. The known constants of the system are:

$R_1 = R_4 = \infty$ $D = 1.000$ $n_o = n_4 = 1$ $n_2 = 1.333$

$R_2 = - R_3 = - 1.000$ $y = 0.500$ $n_1 = n_3 = 1.5170$

With the aid of equations (194) to (202) we find:

$f_1 = - f_2' = + 7.272$ $f_2 = - f_1' = + 5.453$

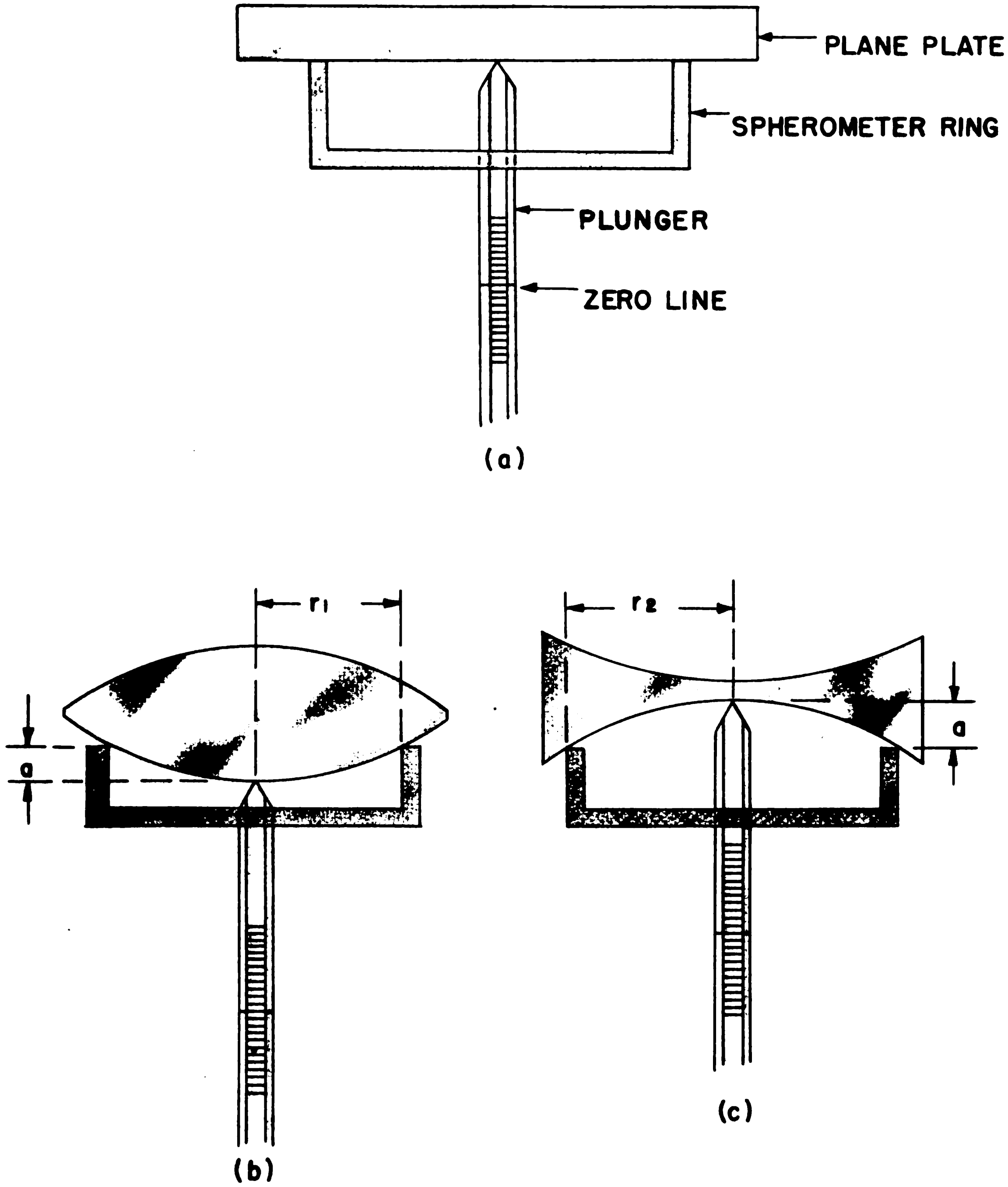


Figure 87. Two thin lenses separated by the distance D.

The remaining constants are found by applying equations (244) to (256).

$f = -f' = 2.928$ $H = -H' = +0.403$ $x = -2.000$ (assumed)

$a = -b = -2.525$ $x' = +4.287$

$s = -4.928$ $s' = +7.215$ $y' = -0.732$

Example 21:

The same two lenses and separation employed in the preceding example will also be used here (see fig. 87(c)); however, the lenses will be in air. The following constants are given:

$R_1 = -R_4 = \infty$ $D = 1.000$ $n_0 = n_2 = n_4 = 1$

$R_2 = -R_3 = -1.000$ $y = 0.500$ $n_1 = n_3 = 1.5170$

Substituting these values first into equations (226) and (232), and then into equations (261) to (268) we find:

$$f_1 = -f_1' = -\frac{R_2}{n_1 - 1} = +1.934 \qquad \phi_1 = -\phi_1' = -(n_1 - 1)\rho_2 = 0.5170$$

$$f_2 = -f_2' = \frac{R_3}{n_3 - 1} = +1.934 \qquad \phi_2 = -\phi_2' = (n_3 - 1)\rho_3 = 0.5170$$

$$f = -f' = \frac{f_1 f_2}{f_1 + f_2 - D} = 1.304 \qquad \phi = \phi' = \phi_1 + \phi_2 - D\phi_1\phi_2 = +0.767$$

$$a = -\frac{(f_2 - D)f_1}{f_1 + f_2 - D} = -0.630 \qquad b = +\frac{(f_1 - D)f_2}{f_1 - f_2 - D} = +0.630$$

$$H = \frac{Df_1}{f_1 + f_2 - D} = +0.674 \qquad H' = -\frac{Df_2}{f_1 + f_2 - D} = -0.674$$

$$x = -1.304 \text{ (assumed)} \qquad x' = -\frac{f^2}{x} = +1.304$$

$$s = f' + x = -2.608 \qquad s' = f + x' = +2.608$$

$$y' = \frac{s'y}{s} = -0.500$$

NOTE: Compare the results obtained in the last two numerical examples. The effect on the various constants of an increase of the refractive index of the medium between the two lenses is also clearly illustrated in the diagrams (b) and (c) of figure 87 which are drawn to the same scale.

Example 22:

The thin collective and the thin dispersive lenses, illustrated in figure 88(a), are separated by a distance of 0.50 inches. The combination, which is in air, shall have a focal length of 2 inches, and the distance between the collective lens and the image plane shall be equal to one inch. The principal ray and the axis include an angle of $-5°$. What are the constants of the lens system?

We have: $f = 2$ inches $L = 1$ inch $s = -\infty$

$D = 0.5$ inches $\mu = -5°$

Substituting these values into equations (276) to (278) we find

$$f_2 = \frac{(D-L)D}{f-L} = -0.250 \qquad f_1 = \frac{(f_2-D)f}{f_2-f} = +0.667$$

$$H = \frac{Df_1}{f_1+f_2-D} = -4.018 \qquad H' = -\frac{Df_2}{f_1+f_2-D} = -1.506$$

$$f = -f' = +2.000 \qquad a = f' + H = -6.018$$

$$b = f + H' = +0.494 \qquad y' = -f \tan \mu = +0.175$$

67. Three Thin Lenses in Contact

The thin lens system shown in figure 88(b) consists of three single lenses, two plane-convex lenses with a bi-convex lens between them. The system is in air and we assume that the focal lengths of the individual lenses are known. In order to compute the focal lengths and the powers of the system we apply the following formulae:

$$f = -f' = \frac{f_1 f_2 f_3}{f_1(f_2+f_3) + f_2 f_3} \tag{279}$$

$$\phi = -\phi' = \phi_1 + \phi_2 + \phi_3 \tag{280}$$

Example 23:

The center lens of the system shown in figure 88(b) has a focal length of 3 inches and each of the outer lenses 4 inches. The object, 0.5 inches high, is placed at a distance of 2.4 inches in front of the system. What are the other constants of the system?

We have, $f_1 = f_3 = 4$ inches $s = -2.40$ inches, and

$f_2 = 3$ inches. $y = +0.50$ inch

Substituting these values into equations (279), (280), (171), (156), (172), and (159), we find

$f = -f' = 1.200$ inches $s' = +2.40$ inches $x = -x' = -1.20$ inches

$\phi = -\phi' = 0.833$ $y' = -0.500$ inch

68. Three Thin Lenses Separated by the Distances D_1 and D_2

In figure 88(c) are shown three thin lenses with air spaces between them and the entire system surrounded by air. If the focal lengths of the individual lenses are known,

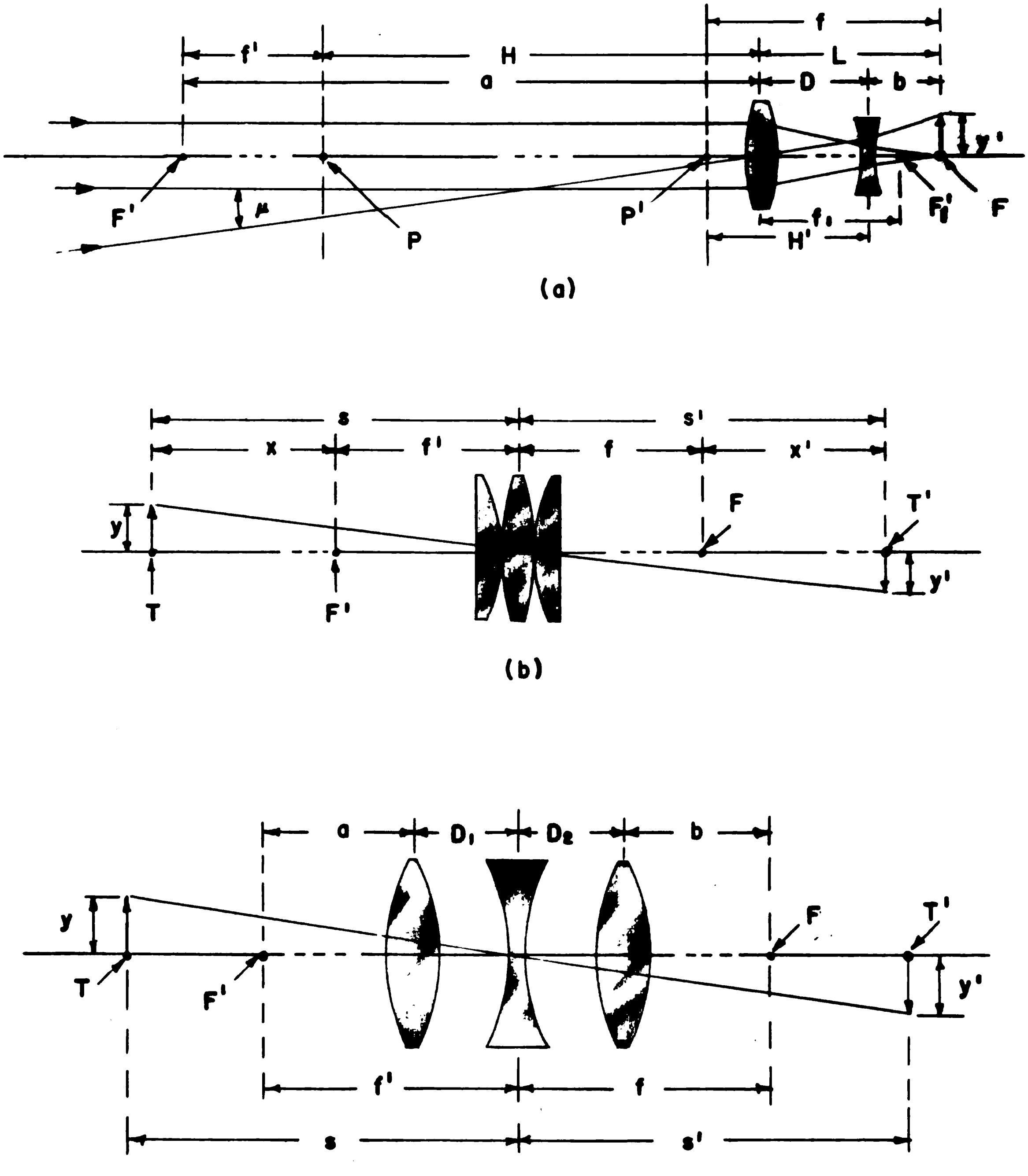


Figure 88. Various thin lens systems.

we determine the remaining constants of the system with the aid of the following set of formulae:

$$f = -f' = \frac{f_1 f_2 f_3}{(f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)} \tag{281}$$

$$\phi = -\phi' = \phi_1 + \phi_2 + \phi_3 - D_1\phi_1(\phi_2 + \phi_3) - D_2\phi_3(\phi_1 + \phi_2) + D_1 D_2 \phi_1 \phi_2 \phi_3 \tag{282}$$

$$a = -\frac{[(f_3 - D_2) f_2 - (f_2 + f_3 - D_2) D_1]\, f_1}{(f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)} = H - f \tag{283}$$

$$b = +\frac{[(f_1 - D_1) f_2 - (f_1 + f_2 - D_1) D_2]\, f_3}{(f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)} = H' + f \tag{284}$$

$$H = +\frac{[f_2 D_2 + (f_2 + f_3 - D_2) D_1]\, f_1}{(f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)} = a - f' \tag{285}$$

$$H' = -\frac{[f_2 D_1 + (f_1 + f_2 - D_1) D_2]\, f_3}{(f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)} = b - f \tag{286}$$

The application of these equations will be demonstrated with the following numerical example:

Example 24:

The focal length of each of the collective lenses of the lens system, shown in figure 88(c) is equal to 3 inches, while that of the dispersive lens is equal to —4 inches. Each collective lens is separated from the dispersive lens by a distance of 0.5 inches. The object, 0.75 inches high, is placed at a distance of 5.236 inches from the first principal point P of the lens system. What are the constants of the combination?

We have:

$f_1 = f_3 = +3$ $\quad$ $D_1 = D_2 = 0.50$ $\quad$ $y = +0.75$

$f_2 = -4$ $\quad$ $s = -5.236$

Substituting these values into equations (281), (283), (285), (171), and (159), we find:

$f = -f' = +2.618$ $\quad$ $a = -b = -2.018$ $\quad$ $s' = -s = +5.236$

$\phi = -\phi' = +0.382$ $\quad$ $H = -H' = +0.60$ $\quad$ $y' = -y = -0.75$

69. Magnification of Distant Objects

The true height of a distant object (see fig. 89(a)) is designated by the letter y. The apparent height, that is, the height under which the object appears to the unaided eye at a distance S, is denoted by the letter a. The distance S is called the "*DISTANCE OF DISTINCT VISION*," or the "*NORMAL READING DISTANCE*." For a normal eye, S = 250 mm or about 10 inches. We can measure the apparent height of the target by holding a scale at a distance of 10 inches before the eye, and in line with the object, and then simply reading off the height.

If we now place the collective lens L in the path of the light, then $T' - T_1'$ is the inverted image of the target $T - T_1$, which can be plainly seen if we put a screen in the image plane. This is illustrated in figure 89 (b), where y' is the height of the image. The image, when compared to the apparent height of the target is magnified M-times, and the value of M is computed as follows:

$$M = \frac{y'}{a} = \frac{f}{S} \tag{287}$$

Example 25:

A collective lens, having a focal length of 30 inches, produces a real inverted image of the object. What are the magnification and the diameter of the image when the apparent height of the object is 5 millimeters?

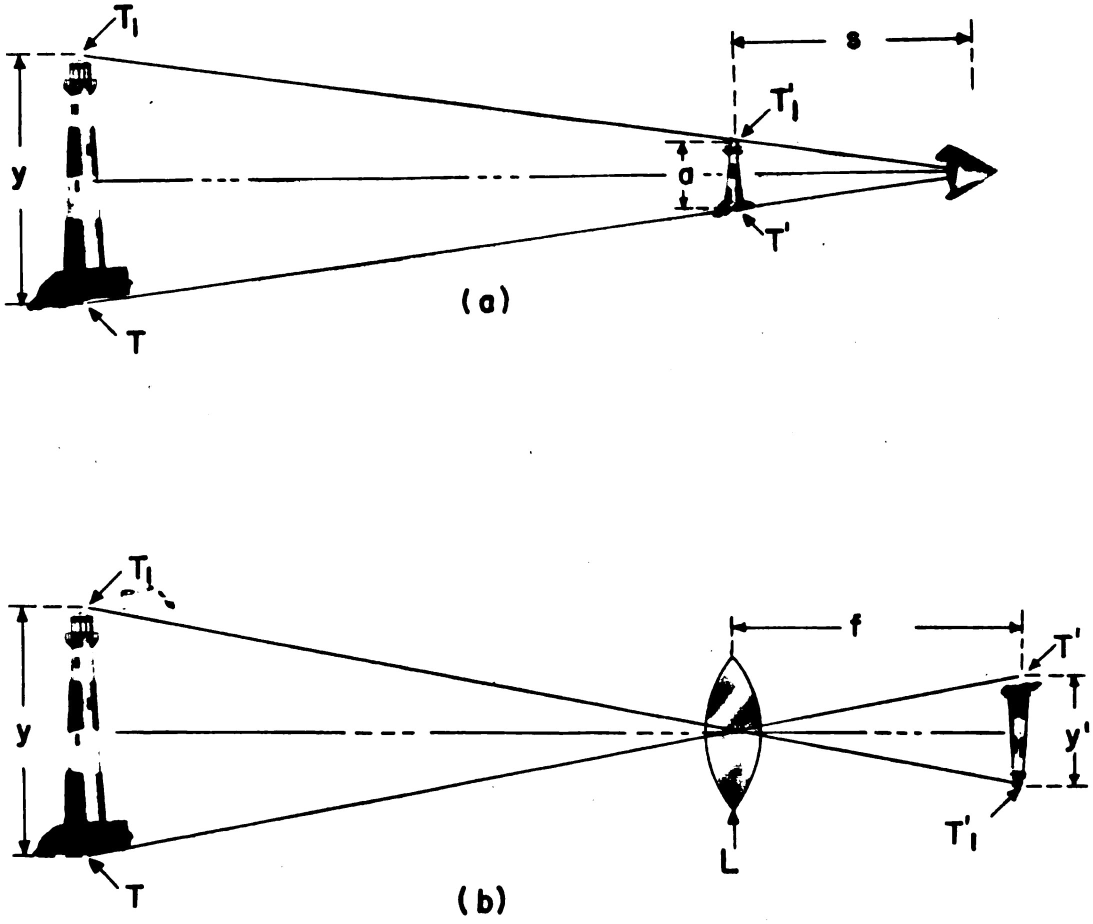


Figure 89. Magnification of distant objects.

We have, $f = 30$ inches, $a = 5$ mm $= 0.197$ inches, and $S = 10$ inches

From equation (287) we obtain, $M = \frac{30}{10} = 3x$ and

$$y' = aM = 0.591 \text{ inches}$$

70. Magnification of Near Objects

Near objects are magnified with the aid of a lens or a lens-system. If the magnification does not exceed the limit of 3x the lens is called a *"READING-GLASS,"* and if the magnification varies between 3x and 20x the lens (or lens-system) is referred to as a *"MAGNIFIER."* Much higher magnifications are obtained with a *"MICROSCOPE."* However, we are here interested only in magnifiers and reading-glasses.

In the case of a magnifier, the object to be observed is placed very close to the first focal plane of the lens, and the observer places his eye about at the rear focal point of the lens. Numerous methods have been used in developing formulae for the determination of the magnification of magnifiers. The methods used by Abbe, Gleichen, and Thompson will be explained in the following paragraphs.

Abbe's method of determining the magnification is illustrated by the diagram (a) in figure 90. The eye is placed behind the magnifier so that its point of rotation will coincide with the second focal point of the magnifier. From the diagram we derive easily

$$\tan \omega = \frac{y}{S} \text{ and } y' = \frac{yS}{f} \text{ and } \tan \omega' = \frac{y}{f} \tag{288}$$

where f is the focal length of the magnifier L, and ω is the angle under which the object appears to the unaided eye, while ω' is the angle under which the target appears to the eye after the rays have passed through L. In this case the magnification is expressed as follows:

$$M = \frac{\tan \omega'}{\tan \omega} = \frac{y'}{y} = \frac{S}{f} = \frac{10}{f} \tag{289}$$

The magnification thus obtained is termed the *"ABSOLUTE MAGNIFICATION"* of the magnifier.

Gleichen's method is illustrated in figure 90(b). In this case the eye is placed behind the magnifier L so that the front focal point of the eye will coincide with the second principal point of the lens. The formulae derived are identical with those given by Abbe. Gleichen added the following equation for computing the absolute magnification:

$$M = \frac{D_p}{4} \tag{290}$$

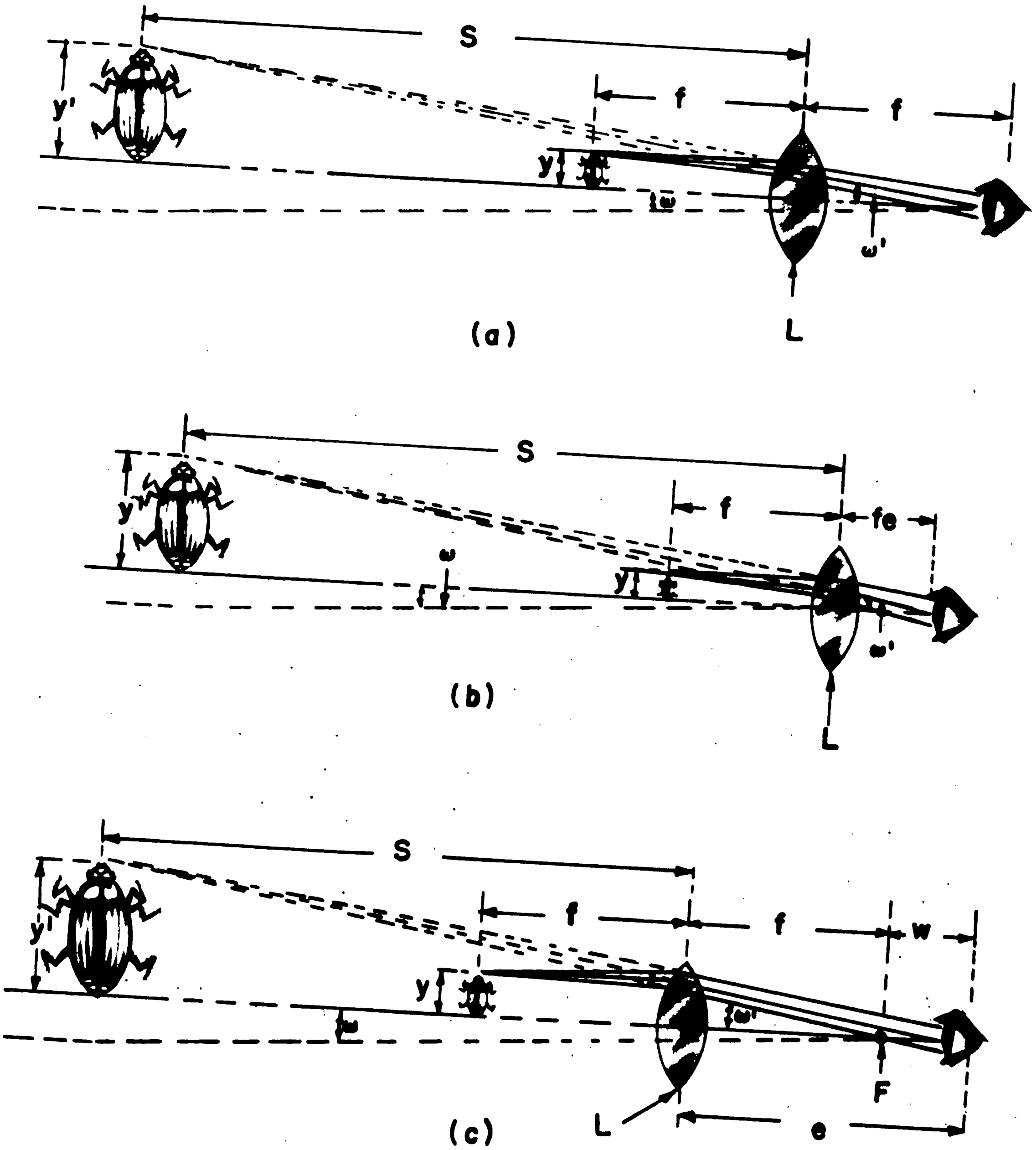


Figure 90. Magnification of near objects.

where D_p is the focal length of the magnifier expressed in diopters. Observers inflicted with either near- or far-sightedness obtain a magnification which is termed the "*INDIVIDUAL NORMAL MAGNIFICATION*"; its magnitude is determined as follows,

$$M' = S\left[\frac{1}{f} - \frac{\Delta}{1000}\right] \tag{291}$$

Here Δ, the refraction error of the eye expressed in diopters, is considered positive in the case of far-sightedness and negative in the case of near-sightedness. The lens focal length f is expressed in millimeters.

The individual normal magnification of a lens in normal position (see fig. 90(b)) is also equal to the sum of D_p and the focal point refraction P of the eye divided by 4, hence,

$$M' = \frac{D_p + P}{4} \tag{292}$$

If the distance between the magnifier L and the eye is increased by an amount w as shown in figure 90(c), the magnification will also increase. From the diagram we derive,

$$\frac{\tan \omega'}{y} = \frac{1}{f}\left\{1 + \frac{w}{S}\right\} \quad \text{but } y = S \tan \omega, \quad \text{hence}$$

$$M = \frac{\tan \omega'}{\tan \omega} = \frac{S + w}{f} \tag{293}$$

Thompson derived the following formulae for the magnification of a lens for any distance:

$$M = \frac{S}{f} + \frac{e - f}{f} \tag{294}$$

where e is the distance from the lens to the point of rotation of the eye. Figure 90(c) will reveal the fact that the difference e — f is equal to the value ω in Gleichen's formulae and, therefore, equations (293) and (294) are identical.

71. Reading Glass

The reading glass, illustrated in figure 91, usually consists of a large single collective lens L. The object to be viewed is placed between the front focal point of the lens and the lens itself, that is, the distance s in front of the lens. The observer will place his eye behind the lens at a much greater distance than that required in the case of a magnifier. The optical constants of a reading glass are computed with the following equations:

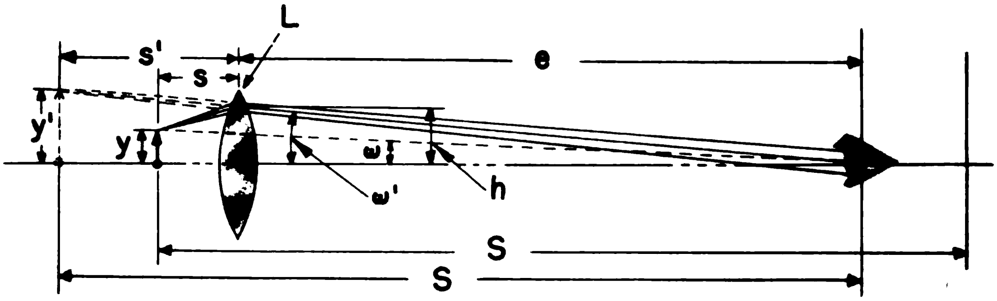


Figure 91. The reading glass.

$$\begin{aligned} \tan \omega &= \frac{y}{S} \\ \tan \omega' &= \frac{y'}{S} \end{aligned} \qquad (295)$$

$$y' = My \qquad (296)$$

$$M = \frac{\tan \omega'}{\tan \omega} = \frac{s'}{s} = \frac{y'}{y} = \frac{f}{f+s} = \frac{e-S}{s} \qquad (297)$$

$$f = \frac{ss'}{s-s'} = \frac{Ms}{1-M} = \frac{e-S}{1-M} \qquad (298)$$

$$s' = \frac{fs}{f+s} \qquad (299)$$

$$e = S + s' = \frac{hS}{y'} \qquad (300)$$

$$h = e \tan \omega' = \frac{ey'}{S} \qquad (301)$$

$$y = \frac{hS}{eM} = \frac{y'}{M} = S \tan \omega \qquad (302)$$

Example 26:

What is the absolute magnification of a magnifier having a focal length of two inches?

We have, $f = 2$ inches $= 20D_p$ and $S = 10$ inches.

Substitute the values into equations (289) and (290) and find,

$$M = \frac{10}{2} = \frac{20}{4} = 5x$$

Example 27:

The focal point refraction of a near-sighted eye is equal to $+5\ D_p$. What is the magnification the observer will obtain with a lens of 50 mm focal length?

We have, $f = 50\ \text{mm} = 20\ D_p \quad \Delta = -5\ D_p \quad P = +5\ D_p \quad S = 250\ \text{mm}$

Applying equations (291) and (292) we find,

$$M = 250 \left\{ \frac{1}{50} + \frac{5}{1000} \right\} = \frac{20+5}{4} = 6.25\text{x}$$

If, on the other hand, the eye is far-sighted by $5\ D_p$, then

$\Delta = +5\ D_p$ and $P = -5\ D_p$, and we have,

$$M = 250 \left\{ \frac{1}{50} - \frac{5}{1000} \right\} = \frac{20-5}{4} = 3.75\text{x}$$

Example 28:

Determine the magnifications produced by a lens of 2 inch focal length for eye distances varying between the limits of 2 and 6 inches.

We have, $f = 2$ inches, $S = 10$ inches, while e assumes various values between 2 and 6. Substitute these values into equation (294) and find:

e =	2	3	4	5	6
M =	5x	5.5x	6x	6.5x	7x

Example 29:

A millimeter scale is placed at a distance of 25 mm in front of a reading-glass. What must be the focal length of the lens when a magnification of 2x is to be obtained when the eye is placed at a distance e behind the lens?

We have, $M = 2\text{x}$, $s = -25$ mm., and $S = 250$ mm.

Substituting these values into equations (298), (299), and (300), we find,

$$f = \frac{Ms}{1-M} = \frac{-50}{-1} = 50\ \text{mm} \qquad s' = \frac{fs}{f+s} = \frac{-1250}{+2s} = -50\ \text{mm}$$

$$e = S + s' = 200\ \text{mm}$$

Example 30:

An object, 20 mm in diameter, is placed at a distance of 50 mm in front of a reading-glass. What are the remaining optical constants if the eye is placed at a distance of 150 mm behind the lens?

We have, $e = 150$ mm $\qquad y = 10$ mm

$s = -50$ mm

Apply equations (297), (296), (298), (299), (301), and find,

$$M = \frac{e - S}{s} = 2x \qquad y' = My = 20 \text{ mm}$$

$$f = \frac{e - S}{1 - M} = 100 \text{ mm} \qquad s' = \frac{fs}{f + s} = -100 \text{ mm}$$

$$h = \frac{ey'}{S} = 12 \text{ mm}$$

2h = 24 mm is the minimum aperture we can give the lens if 20 mm on the scale are to be seen by the eye at rest on the axis.

PART 4. THICK LENS

72. General Formulae for Thick Lenses

These abbreviations will be used in the following formulae:

$$B = n_1(n_1 - n_o)R_2 + (n_2 - n_1)[n_1R_1 - (n_1 - n_o)d] \tag{303}$$

$$E = n_1(R_2 - R_1) + (n_1 - 1)d \tag{304}$$

$$G = (n_1 - 1)E \tag{305}$$

$$f = +\frac{n_1n_2R_1R_2}{B} \tag{306}$$

$$f' = -\frac{n_on_1R_1R_2}{B} = -\frac{n_of}{n_2} \tag{307}$$

$$\text{F.F.L.} = -\frac{n_oR_1[n_1R_2 - (n_2 - n_1)d]}{B} \tag{308}$$

$$\text{B.F.L.} = +\frac{n_2R_2[n_1R_1 - (n_1 - n_o)d]}{B} \tag{309}$$

$$H = +\frac{n_o(n_2 - n_1)dR_1}{B} \tag{310}$$

$$H' = -\frac{n_2(n_1 - n_o)dR_2}{B} \tag{311}$$

$$K = \frac{[n_1(n_2 - n_o)R_2 + n_o(n_2 - n_1)d]\,R_1}{B} \tag{312}$$

$$K' = \frac{[n_1(n_2 - n_o)R_1 - n_2(n_1 - n_o)d]\,R_2}{B} \tag{313}$$

$$H'' = K'' = \frac{(n_1 - n_o)(n_2 - n_1)(R_1 - R_2 - d)d}{B} \tag{314}$$

$$\frac{H'}{H} = -\frac{n_2(n_1 - n_o)R_2}{n_o(n_2 - n_1)R_1} \quad \text{(CHECK FORMULA)} \tag{315}$$

When the refractive indices n_v are given the following equations hold:

If d, f, and R_2 are known, then

$$R_1 = \frac{(n_1 - n_o)[n_1R_2 - (n_2 - n_1)d]\,f}{n_1[n_2R_2 - (n_2 - n_1)f]} \tag{316}$$

If d, f, and R_1 are known, then

$$R_2 = \frac{(n_2 - n_1)[n_1R_1 - (n_1 - n_o)d]\,f}{n_1[n_2R_1 - (n_1 - n_o)f]} \tag{317}$$

If f, R_1 and R_2 are known, then

$$d = \frac{n_1[(n_2 - n_1)R_1 + (n_1 - n_o)R_2]\,f - n_1n_2R_1R_2}{(n_1 - n_o)(n_2 - n_1)f} \tag{318}$$

In the following set of equations it is assumed that the lenses are in air:

$$n_o = n_2 = 1$$

$$f = -f' = \frac{n_1R_1R_2}{G} \tag{319}$$

$$\text{F.F.L.} = -\frac{[n_1R_2 + (n_1 - 1)d]\,R_1}{G} = f' + H \tag{320}$$

$$\text{B.F.L.} = +\frac{[n_1R_1 - (n_1 - 1)d]\,R_2}{G} = f + H' \tag{321}$$

$$H = -\frac{dR_1}{E} \tag{322}$$

$$H' = -\frac{dR_2}{E} \tag{323}$$

$$H'' = \frac{(n_1 - 1)(R_2 - R_1 + d)d}{E} \tag{324}$$

$$\frac{H'}{H} = \frac{R_2}{R_1} \tag{325}$$

If the refractive index n_1 is given, and

d, f, and R_2 are known, then

$$R_1 = \frac{(n_1 - 1)[n_1 R_2 + (n_1 - 1)d]f}{n_1[R_2 + (n_1 - 1)f]} \tag{326}$$

d, f and R_1 are known, then

$$R_2 = -\frac{(n_1 - 1)[n_1 R_1 - (n_1 - 1)d]f}{n_1[R_1 - (n_1 - 1)f]} \tag{327}$$

f, R_1 and R_2 are known, then

$$d = \frac{n_1[R_1 R_2 - (n_1 - 1)(R_2 - R_1)f]}{(n_1 - 1)^2 f} \tag{328}$$

Example 31:

The collective lens shown in figure 92(a) is surrounded by water. Its radii are + 1.250 inches and − 0.750 inches, respectively. The axial thickness of the lens is 0.500 inches. The object, 0.500 inch high, is located at a distance of − 7.020 inches from the first principal plane of the lens. What are the magnitudes of the unknown constants?

We have $R_1 = +1.250$ $y = +0.500$ $n_0 = n_2 = 1.333$

$R_2 = -0.750$ $s = -7.020$ $n_1 = 1.5165$

$d = 0.500$

Substituting these values into equations (303), (306) to (315), (153), (150), (151), and (159), we find,

$B = -0.540$ $H = K = +0.283$ $s' = -s = +7.020$

$f = -f' = +3.510$ $H' = K' = -0.170$ $x = -x' = -3.510$

$\text{F.F.L.} = -3.226$ $H'' = K'' = +0.047$ $y' = -0.500$

$\text{B.F.L.} = +3.340$ $H'/H = -0.60$

Example 32:

What are the values of the lens constants considered in the preceding example if the lens is in air? We have:

$R_1 = +1.250$ $y = +0.500$ $n_0 = n_2 = 1.000$

$R_2 = -0.750$ $s = -1.9840$ $n_1 = 1.5165$

$d = 0.500$

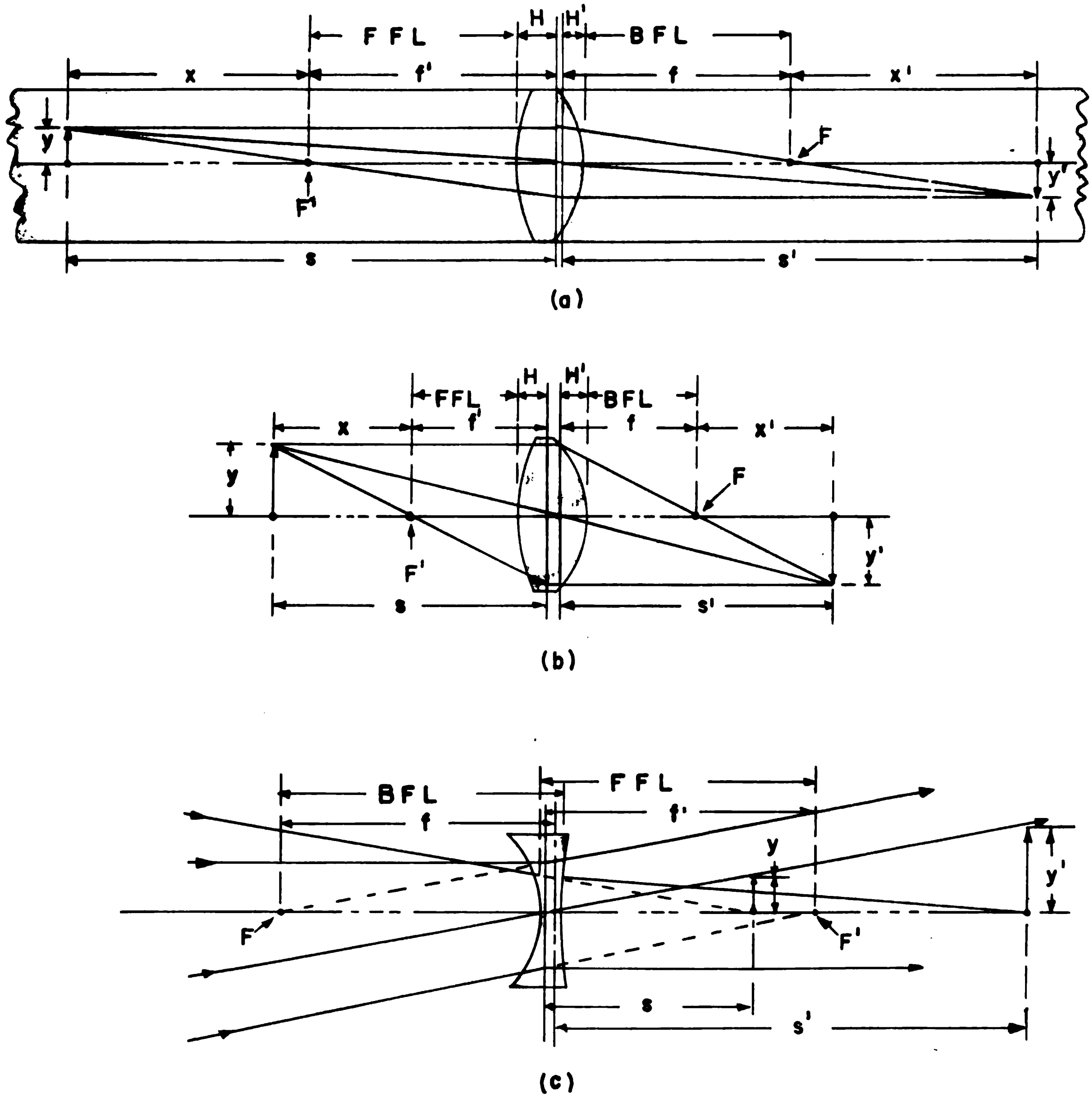


Figure 92. Construction of the image.

Apply equations (304), (305), (319) to (325), (171), (150), (151), and (159) and find:

$G = -1.433$

$f = -f' = +0.992$

$F.F.L. = -0.767$

$B.F.L. = +0.857$

$H = +0.225$

$H' = -0.135$

$H'' = +0.140$

$H'/H = -0.600$

$s' = -s = +1.984$

$x = -x' = -0.992$

$y' = -0.500$

These dimensions are shown in figure 92(b).

Example 33:

The dispersive lens shown in figure 92(c) has a focal length of − 2.000 inches in air, the axial thickness is equal to 0.150 inches, and the first radius is − 1.750. The object, 0.250 inch high, is at a distance of 1.500 inches to the right of the second principal plane of the lens. What are the magnitudes of the unknown constants?

We have:

$f = -f' = -2.000$

$R_1 = -1.750$

$d = 0.150$

$s = +1.500$

$y = +0.500$

$n_0 = n_2 = 1.000$

$n_1 = 1.6490$

Apply equations (327), (320) to (325), (153), (159), and we find,

$R_2 = +5.195$

$E = +11.550$

$G = +7.496$

$F.F.L. = +2.023$

$B.F.L. = -2.067$

$H = +0.023$

$H' = -0.068$

$H'' = +0.060$

$H'/H = -2.969$

$s' = +6.000$

$y' = +1.000$

The construction of the image formed by a thick single lens is also illustrated in the diagrams of figure 92.

73. Special Formulae for Thick Lenses

In many cases it is desired that the lens have a special form. We simplify the formulae (306) to (328) for the various cases. In fire control instruments the lenses are invariably in air, hence, we limit our formulae to cases so that $n_0 = n_2 = 1$.

The Lens is Either Equi Bi-Convex or Equi Bi-Concave

$$f = -f' = \frac{n_1 R_1^2}{(n_1 - 1)\,[2n_1R_1 - (n_1 - 1)d]} \tag{329}$$

$$F.F.L. = -B.F.L. = -\frac{[n_1R_1 - (n_1 - 1)d]\,R_1}{(n_1 - 1)\,[2n_1R_1 - (n_1 - 1)d]} \tag{330}$$

$$H = -H' = +\frac{dR_1}{2n_1R_1 - (n_1 - 1)d} \tag{331}$$

$$H'' = \frac{(n_1 - 1)(2R_1 - d)d}{2n_1R_1 - (n_1 - 1)d} \qquad (332)$$

$$\frac{H'}{H} = -1 \qquad \text{(CHECK FORMULA)} \qquad (333)$$

When d, f, and n_1 are known then,

$$R_1 = -R_2 = \frac{(n_1 - 1)[n_1f \pm \sqrt{n_1f(n_1f - d)}]}{n_1} \qquad (334)$$

When f, n_1, and R_1 are known then,

$$d = \frac{n_1R_1 [2(n_1 - 1)f - R_1]}{(n_1 - 1)^2 f} \qquad (335)$$

$R_2 = rR_1$

$$f = -f' = \frac{n_1rR_1^2}{(n_1 - 1)[n_1(r - 1)R_1 + (n_1 - 1)d]} \qquad (336)$$

$$\text{F.F.L.} = -\frac{[n_1rR_1 + (n_1 - 1)d]\, R_1}{(n_1 - 1)[n_1(r - 1)R_1 + (n_1 - 1)d]} \qquad (337)$$

$$\text{B.F.L.} = +\frac{[n_1R_1 - (n_1 - 1)d]\, rR_1}{(n_1 - 1)[n_1(r - 1)R_1 + (n_1 - 1)d]} \qquad (338)$$

$$H = -\frac{dR_1}{n_1(r - 1)R_1 + (n_1 - 1)d} \qquad (339)$$

$$H' = -\frac{drR_1}{n_1(r - 1)R_1 + (n_1 - 1)d} \qquad (340)$$

$$H'' - \frac{(n_1 - 1)d\,[(r - 1)R_1 + d]}{n_1(r - 1)R_1 + (n_1 - 1)d} \qquad (341)$$

$$H'/H = r \qquad \text{(CHECK FORMULA)} \qquad (342)$$

When d, f, n, and r are known then,

$$R_1 = \frac{(n_1 - 1)[z \pm \sqrt{z^2 + 4\,n_1dfr}}{2n_1r} \qquad (343)$$

where $z = n_1(r - 1)f$, and $R_2 = rR_1$

When f, n, r, and R_1 are known then,

$$d = \frac{n_1R_1 [nR_1 - (n_1 - 1)(r - 1)f]}{(n_1 - 1)^2 f} \qquad (344)$$

$R_1 = \pm \infty$, R_2 is Either Positive or Negative

$$f = -f' = -\frac{R_2}{n_1 - 1} \tag{345}$$

$$\text{F.F.L.} = \frac{n_1 R_2 + (n_1 - 1)d}{n_1(n_1 - 1)} \tag{346}$$

$$\text{B.F.L.} = f \tag{347}$$

$$H = \frac{d}{n_1} \tag{348}$$

$$H' = \pm 0 \tag{349}$$

$$H'' = \frac{(n_1 - 1)d}{n_1} \tag{350}$$

When f and n_1 are known then,

$$R_2 = -(n_1 - 1)f \tag{351}$$

$R_2 = \pm \infty$, R_1 is Either Positive or Negative

$$f = -f' = \frac{R_1}{n_1 - 1} \tag{352}$$

$$\text{F.F.L.} = -f \tag{353}$$

$$\text{B.F.L.} = \frac{n_1 R_1 - (n_1 - 1)d}{n_1(n_1 - 1)} \tag{354}$$

$$H = \pm 0 \tag{355}$$

$$H' = -\frac{d}{n_1} \tag{356}$$

When f and n_1 are known then,

$$R_1 = (n_1 - 1)f \tag{357}$$

The Concentric Lens

In this case the centers of curvature C of the two surfaces coincide, hence:

$$R_2 = R_1 - d \tag{358}$$

This will be readily understood by a glance at figure 88. The various constants of this lens are determined with the aid of the following equations:

$$f = -f' = \frac{n_1 R_1 R_2}{(n_1 - 1)(R_2 - R_1)} \tag{359}$$

$$\text{F.F.L.} = -\frac{[(n_1-1)R_1+R_2]\,R_1}{(n_1-1)(R_2-R_1)} \tag{360}$$

$$\text{B.F.L.} = +\frac{[(n_1-1)R_2+R_1]\,R_2}{(n_1-1)(R_2-R_1)} \tag{361}$$

$$H = R_1 \tag{362}$$

$$H' = R_2 \tag{363}$$

$$d = R_1 - R_2 \tag{364}$$

When d, f and n_1 are known then,

$$R_1 = \frac{n_1 d \pm \sqrt{n_1 d\,[n_1 d - 4(n_1-1)f]}}{2n_1} \tag{365}$$

The Sphere

In the case of a sphere, the following relations exist:

$R_2 = -R_1$ and $d = 2R_1$ The various constants are determined as follows:

$$f = -f' = \frac{n_1 R_1}{2(n_1-1)} \tag{366}$$

$$\text{F.F.L.} = -\,\text{B.F.L.} = -\frac{(2-n_1)R_1}{2(n_1-1)} \tag{367}$$

$$H = -H' = R_1 \tag{368}$$

When f and n_1 are given then,

$$R_1 = -R_2 = \frac{2(n_1-1)f}{n_1} \tag{369}$$

$$d = 2R_1 \tag{370}$$

Example 34:

The equi bi-convex lens shown in figure 93(a) has a focal length of 1.500 inches and the thickness of the lens = 0.300 inches. What are the various constants of the lens when the refractive index is equal to 1.5170?

We have: $f = 1.500$, $d = 0.300$, and $n_1 = 1.5170$

Substituting these values into equation (334) and then into (329) to (333) we find,

$f = -f' = +1.500$ F.F.L. = − B.F.L. = − 1.398

$R_1 = -R_2 = +1.498$ $H = -H' = +0.102$

$H'' = 0.095$

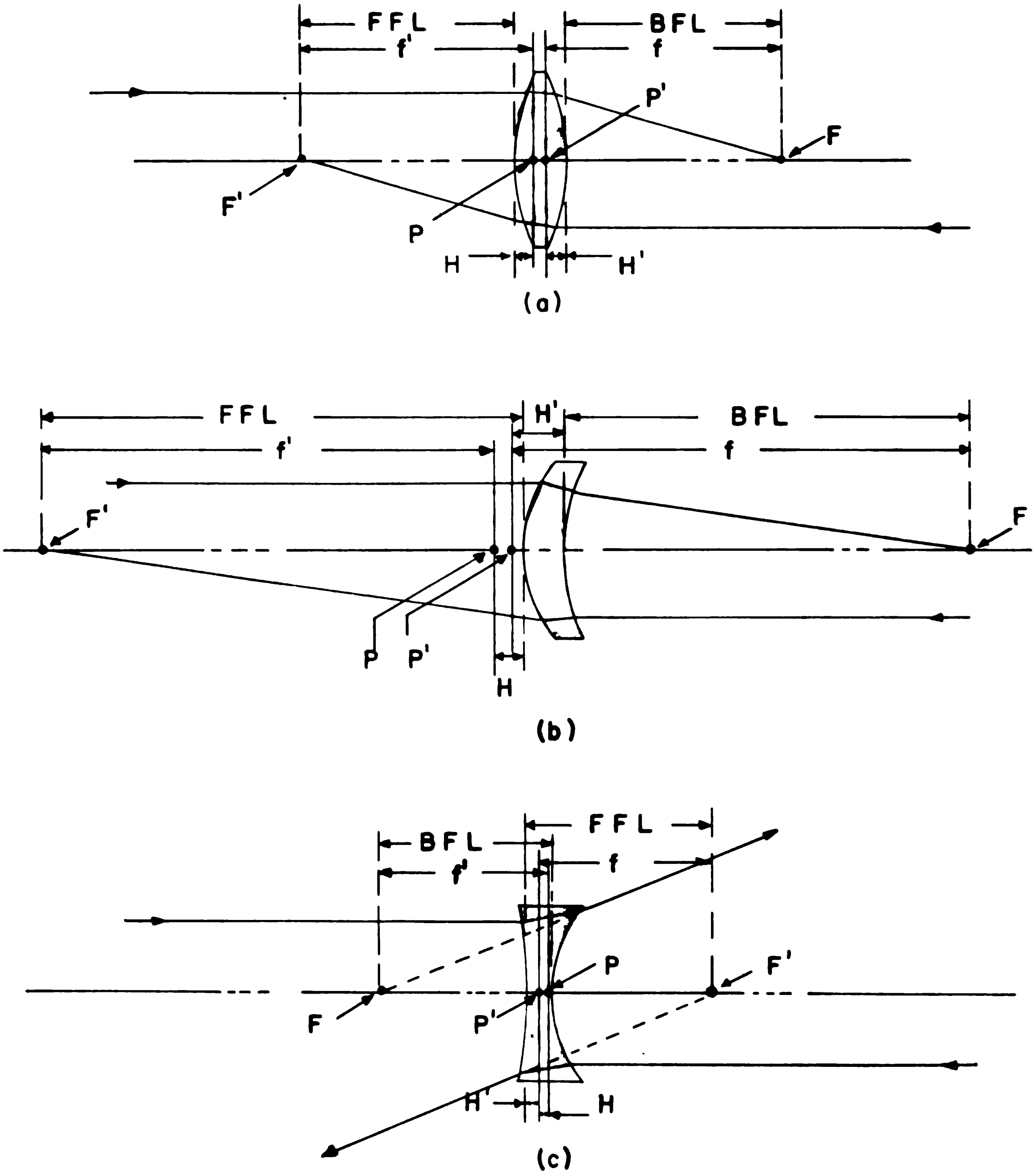


Figure 93. The constants of thick lenses.

$$\left.\begin{aligned} -f' + \text{F.F.L.} = \text{B.F.L.} - f = H = -H' = 0.102 \\ d - 2H = H'' = 0.096 \\ d = 0.300, \text{ by equation (335)} \end{aligned}\right\} \text{CHECK}$$

Example 35:

The positive meniscus lens shown in figure 93(b) has a focal length of 3.000 inches, and the radii shall be in a ratio of 1 to 2. The lens has a thickness of 0.300 inches and the refractive index is 1.5170. What are the various constants of the lens?

We have, $f = -f' = 3.000$ $d = 0.300$ $n_1 = 1.5170$ $r = \frac{R_2}{R_1} = 2$

Apply first equation (343) and then equations (336) to (342).

$R_1 = +0.867$ F.F.L. $= -3.177$ $H = -0.177$ $H'' = +0.123$

$R_2 = +1.734$ B.F.L. $= +2.646$ $H' = -0.354$

Example 36:

The radii of the dispersive lens illustrated in figure 93(c) are − 3.000 inches and + 1.000 inch, respectively. If the refractive index of the glass is 1.6490, what must be the axial thickness of the lens if its focal length is to equal − 1.135 inches?

We have, $R_1 = -3.000$ $R_2 = +1.000$ $f = -f' = -1.135$ $n_1 = 1.6490$

$r = -\frac{R_2}{R_1} = -0.333$. Now apply equations (344) to (350) and find

$d = 0.187$ F.F.L. $= +1.219$ $H = +0.084$

B.F.L. $= -1.163$ $H' = -0.028$

$$\left.\begin{aligned} -f' + \text{F.F.L.} = H = +0.084 \\ -\text{B.F.L.} + f = H' = -0.028 \\ H'/H = r = -0.333 \end{aligned}\right\} \text{CHECK}$$

The various points are illustrated in figure 93(c).

Example 37:

The plane-convex lens shown in figure 94(a) has an axial thickness of 0.250 inches. The refractive index of the glass is 1.5170. What must be the radius of curvature of the convex surface if the lens has a focal length of + 2 inches?

We have, $R_1 = \pm\infty$ $d = 0.250$ and $n_1 = 1.5170$

Required: R_2 for $f_2 + 2$ inches

Substitute the f and n_1 values into equation (351) and find

$R_2 = -1.034$ inches. The remaining constants are computed with equations (345) to (350) and we get:

$f = -f' = +2.000$ F.F.L. $= -1.835$ $H = +0.165$

B.F.L. $= +2.000$ $H' = \pm 0$

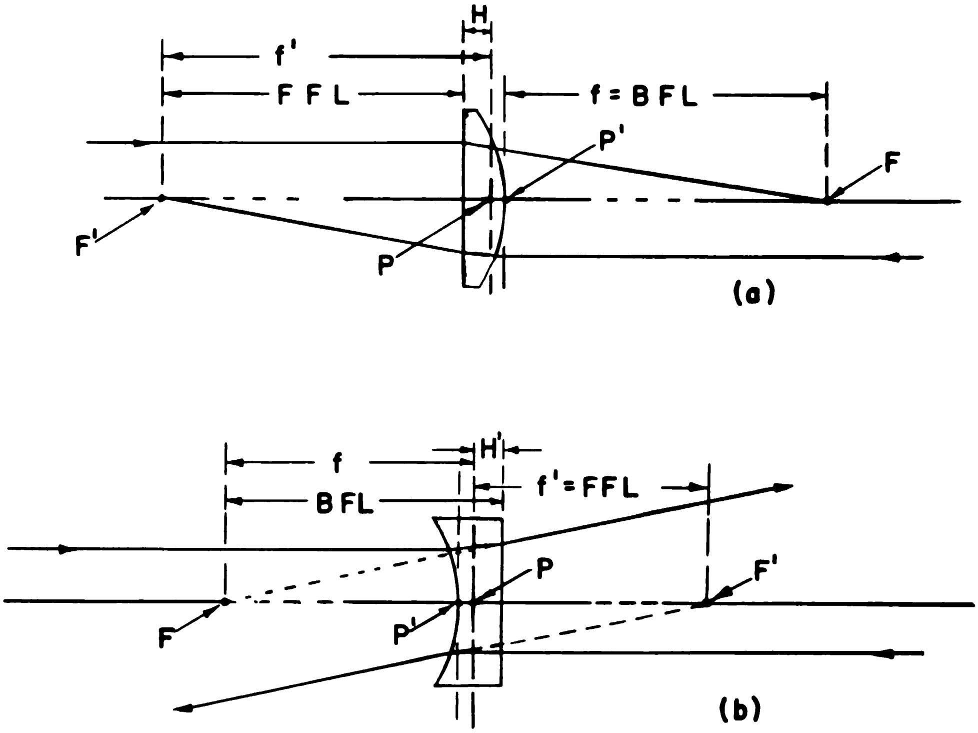


Figure 94. The constants of thick lenses.

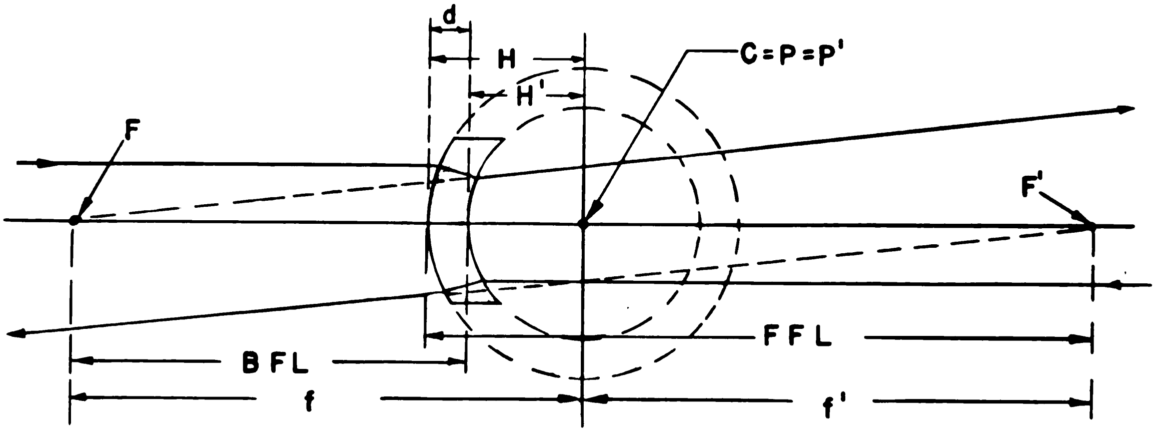


Figure 95. The concentric lens.

Example 38:

In figure 94(b) is shown a concave-plane lens. The concave surface has a radius of curvature of − 1.000 inch and the refractive index of the glass is 1.6490. What are the constants of the lens when its axial thickness is 0.250 inches?

We have: $R_1 = -1.000$ $R_2 = \pm \infty$ $d = 0.250$ $n_1 = 1.6490$

Substitute these values into equations (352) to (356) and find:

$f = -f' = -1.541$ F.F.L. $= +1.541$ $H = \pm 0$

B.F.L. $= -1.692$ $H' = -0.152$

Example 39:

R_1 and R_2 of the concentric lens shown in figure 95 are + 0.500 and + 0.350 inches, respectively, and the refractive index of the glass is 1.5170. What are the focal lengths and thickness of the lens, and what are the locations of the two principal planes?

We have: $R_1 = +0.500$ $R_2 = +0.350$ $n_1 = 1.5170$

With equation (364) we find, $d = 0.15$ inches

Substituting these values into equations (359) to (363) we obtain the following results:

$f = f' = -3.423$ F.F.L. $= +3.923$ $H = +0.500$

B.F.L. $= -3.073$ $H' = +0.350$

It will be noted that the centers C not only coincide with one another, but they also coincide with the two Gauss points P and P′.

Example 40:

The cross-section of a sphere is shown in figure 96. The diameter of the sphere is 2.5 inches, hence, $d = 2.5$ and $R_1 = -R_2 = 1.25$ inches. Determine the magnitudes of the focal lengths and the position of the principal points (Gauss points), assuming that the refractive index of the glass is 1.5170.

We have: $R_1 = -R_2 = +1.250$ $d = 2.500$ $n_1 = 1.5170$

Apply equations (366) to (368) and find,

$f = -f' = +1.834$ F.F.L. $= -$ B.F.L. $= -0.584$ $H = -H' = +1.250$

Both principal points coincide with the geometrical center of the sphere.

74. Variation in Thickness and its Effect on Power of Lens

The three lenses shown in the diagrams of figure 97 have the following constants in common:

$R_1 = +1.250$ $R_2 = +1.000$ $n_1 = 1.5170$

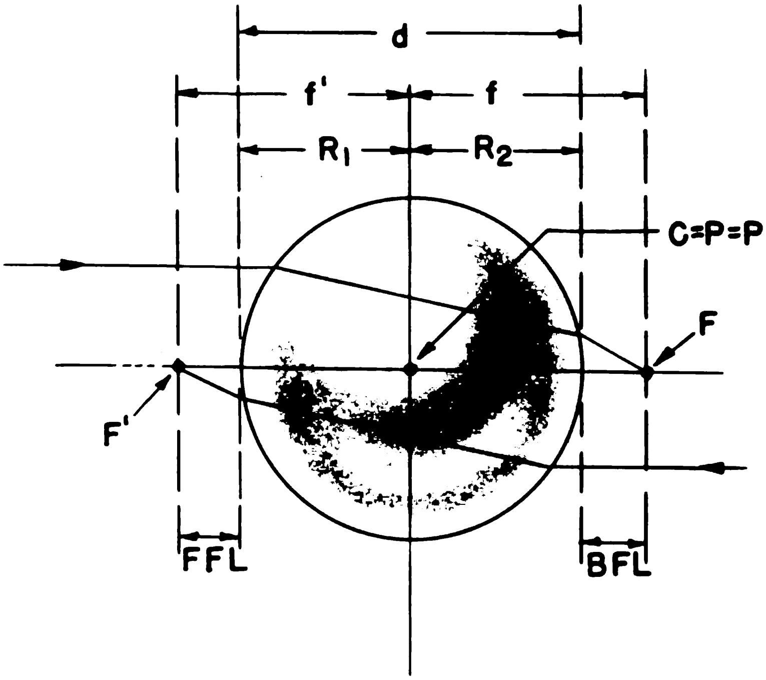


Figure 96. The sphere.

Whether this lens is a collective lens, a telescopic lens, or a dispersive lens depends on whether the thickness of the lens is greater than, equal to, or smaller than the constant Q, whose value is

$$Q = \frac{n_1(R_1 - R_2)}{n_1 - 1} = 0.734 \tag{371}$$

Case 1 *Let $d = 2Q = 1.468$*

Applying equations (319) to (323) we find,

$f = -f' = +9.660$ F.F.L. $= -14.492$ $H = -4.833$

B.F.L. $= +5.793$ $H' = -3.866$

Since the focal length f is positive, the lens is a *"COLLECTIVE LENS"* (see fig. 97(a)).

Case 2 *Let $d = Q = 0.734$*

In this case we find that the values of f, f′, F.F.L., B.F.L., H, and H′ become infinitely large, hence, this lens will act as a plane parallel plate. This lens is referred to as a *"TELESCOPIC LENS,"* or a lens of *"ZERO POWER."* Lenses of this type are often placed behind an objective. This will not alter the focal length of the system but will help to flatten the field (see fig. 97(b)).

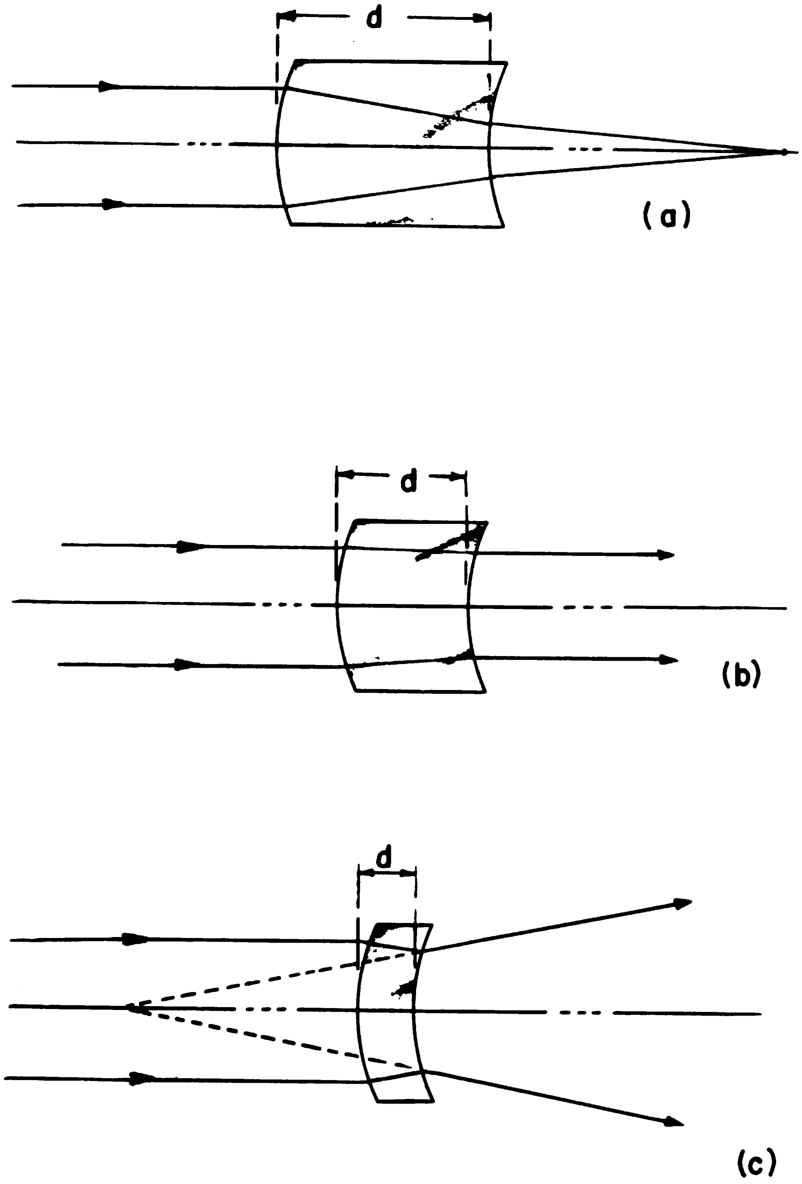


Figure 97. Variation in thickness and its effect on the power of the lens.

Case 3 *Let* $d = Q/2 = 0.367$

In this case we find,

$f = -f' = -19.354$ F.F.L. $= +21.775$ $H = +2.421$

B.F.L. $= -17.417$ $H' = +1.937$

Since the focal length f is negative, this lens is a "*DISPERSIVE LENS*" (see fig. 97(c)).

75. Diameter and Thickness of a Lens

The optical designer and the draftsman frequently require the axial thickness and the edge thickness of a lens. The lens grinder will want to know the diameter of a lens when the same is ground to knife-edge, that is, the lens has no edge thickness. The magnitudes of these constants can easily be computed with the aid of the formulae given in the following:

Abbreviations

a_v = the height of the sagitta of the vth surface

d = the axial thickness of the lens

E.T. = the edge thickness of the lens

2r = the diameter of the lens

R_v = the radius of curvature of the vth lens

NOTE: In order to simplify the computations for the men less skilled in mathematics, it is assumed that all radii, whether convex or concave towards the incident light are positive, hence, the values of a_1 and a_2 will also be positive.

Formulae

$$\begin{aligned} a_1 &= R_1 - \sqrt{R_1^2 - r^2} \\ a_2 &= R_2 - \sqrt{R_2^2 - r^2} \end{aligned} \tag{372}$$

The letter to the left of the following equations corresponds to the letter underneath the diagram of figure 98 to which the formula refers.

Plane Convex Lens

$$\text{(a)} \quad \begin{aligned} d &= E.T. + a_2 \\ E.T. &= d - a_2 \end{aligned} \tag{373}$$

Plane Concave Lens

$$\text{(b)} \quad \begin{aligned} d &= E.T. - a_2 \\ E.T. &= d + a_2 \end{aligned} \tag{374}$$

Bi-Convex Lens

$$\text{(c)} \quad \begin{aligned} d &= E.T. + a_1 + a_2 \\ E.T. &= d - a_1 - a_2 \end{aligned} \tag{375}$$

Bi-Concave Lens

$$\text{(d)} \quad \begin{aligned} d &= E.T. - a_1 - a_2 \\ E.T. &= d + a_1 + a_2 \end{aligned} \tag{376}$$

Positive Meniscus

$$\text{(e)} \quad \begin{aligned} d &= E.T. + a_1 - a_2 \\ E.T. &= d - a_1 + a_2 \end{aligned} \tag{377}$$

Negative Meniscus

$$\text{(f)} \quad \begin{aligned} d &= E.T. = a_1 + a_2 \\ E.T. &= d + a_1 - a_2 \end{aligned} \tag{378}$$

The Lens is Ground to Knife-Edge

In this case E.T. $= 0$

Plane Convex Lens

$$\text{(g)} \quad \begin{gathered} d = a_2 \\ \text{Diameter} = 2r = 2\sqrt{d(2R_2 - d)} \end{gathered} \tag{379}$$

Positive Meniscus

$$\text{(h)} \quad \begin{gathered} d = a_1 - a_2 \\ \text{Diameter} = \frac{\sqrt{(2R_1 - d)d\,\{4(d - R_1 + R_2)R_2 - (2R_1 - d)d\}}}{d - R_1 + R_2} \end{gathered} \tag{380}$$

Bi-Convex Lens

$$\text{(i)} \quad \begin{gathered} d = a_1 + a_2 \\ \text{Diameter} = \frac{\sqrt{(2R_1 - d)d\,\{4(R_1 + R_2 - d)R_2 - (2R_1 - d)d\}}}{R_1 + R_2 - d} \end{gathered} \tag{381}$$

From table XII can be obtained the values of a in a minimum amount of time. Its application will be explained in the following examples. The diagrams for the various examples are shown in figure 98.

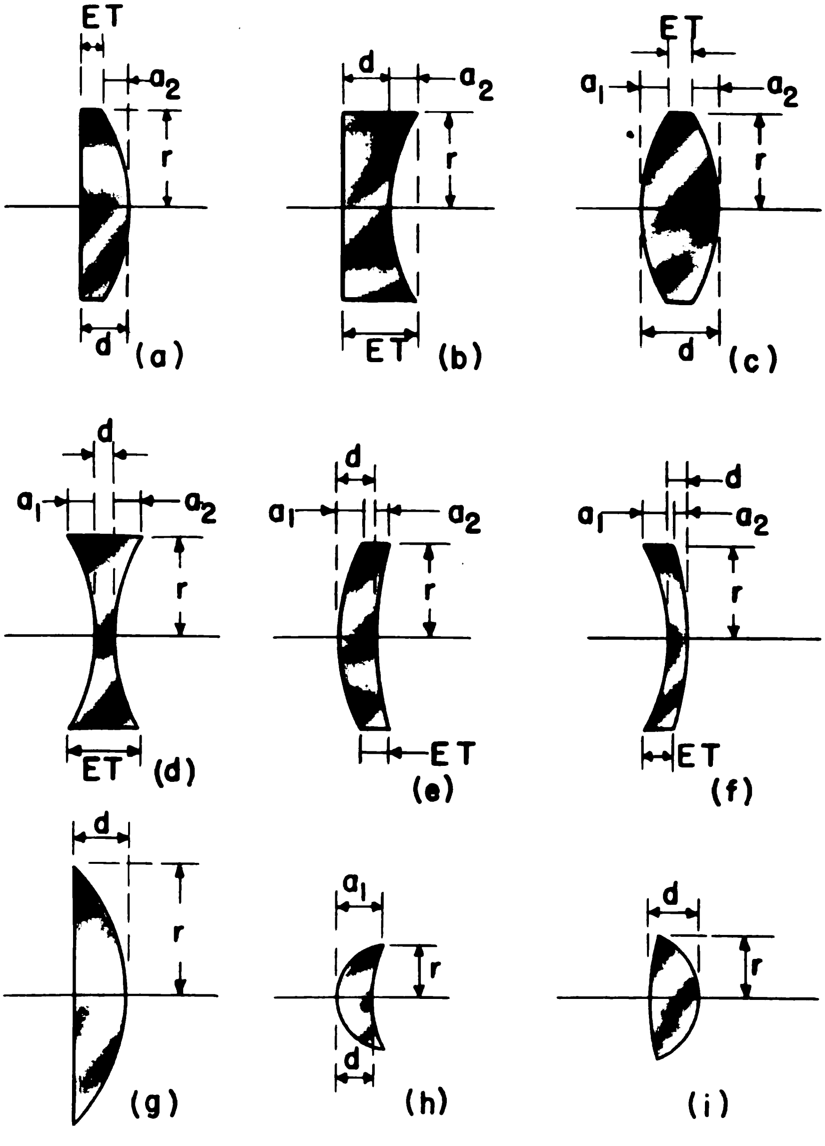


Figure 98. The diameter and the thickness of a lens.

Example 41:

The plane-convex lens shown in diagram (a) has a diameter of one inch and the radius is also one inch. What must be the axial thickness of the lens if an edge thickness of 0.10 inch is required?

We have: $R_1 = \pm \infty$ Diameter $= 1.00$

$R_2 = 1.00$ E.T. $= 0.10$ $d = ?$

DIAMETER/R $=1$, find this value in the first column of table XII. In the second column of the same line we find $a/R = 0.13397$. Multiplying this value by R_2 we obtain $a = 0.13397$. Substitute this value into equation (373) and find, $d = a + \text{E.T.} = 0.23397$.

Example 42:

The radius of the spherical surface of the plane-concave lens shown in diagram (b) is 1.15 inches and the diameter of the lens is 0.75 inches. What is the edge thickness when the axial thickness is equal to 0.10 inch?

We have, $R_1 = \pm \infty$ $d = 0.10$

$R_2 = 1.15$ Diameter $= 0.75$ E.T. $= ?$

DIA/R $= 0.6521738$ and from table XII we find

$a/R_2 = 0.05428$ for $\text{DIA}/R_2 = 0.6521739$, and $a/R_2 = 0.05601$ for

$\text{DIA}/R_2 = 0.660$. Hence, the difference is 0.00173 which is given in the third column of

table XII. Therefore for $\text{DIA}/R_2 = 0.6521739$

$a_2/R_2 = 0.05428 + (.00173 \times 0.21739) = 0.05428 + 0.00038 = 0.05466$

$a_2 = 0.05466$ $R_2 = 0.06286$. Finally, from equation (374)

$\text{E.T.} = a_2 + d = a_2 + d = 0.1629$ inches

Example 43:

The diameter, as well as both radii of the equi bi-convex lens shown in diagram (c) are equal to 1 inch. What must be the axial thickness if the edge thickness is to be 0.10 inches?

We have, $R_1 = R_2 = 1$, Dia. $= 1$, E.T. $= 0.10$ $d = ?$

DIA/R $= 1.000$, and from table XII $a_1/R_1 = a_2/R_2 = 0.13397$, hence from equation (375) $d = a_1 + a_2 + \text{E.T.} = 0.3679$ inches.

Example 44:

The radii and diameter of the equi bi-concave lens shown in diagram (d) are 1 inch. What is the edge thickness of the lens when an axial thickness of 0.10 inches is required?

We have, $R_1 = R_2 = 1.00$, Dia. $=1.00$, $d = 0.10$, E.T. $= ?$

DIA/R $= 1.00$ from table XII, $a_1 = a_2 = 0.13397$, and from equation (376)

$\text{E.T.} = a_1 + a_2 + d = 0.3679$ inches.

TABLE XII

Table for Computing Edge Thickness

DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.
0.000	0.00000	0.00001	0.200	0.00500	0.00052	0.400	0.02020	0.00104	0.600	0.04606	0.00158	0.800	0.08348	0.00219
0.010	0.00001	0.00004	0.210	0.00552	0.00053	0.410	0.02124	0.00105	0.610	0.04764	0.00161	0.810	0.08567	0.00224
0.020	0.00005	0.00006	0.220	0.00605	0.00058	0.420	0.02229	0.00109	0.620	0.04925	0.00165	0.820	0.08791	0.00226
0.030	0.00011	0.00009	0.230	0.00663	0.00059	0.430	0.02338	0.00112	0.630	0.05090	0.00168	0.830	0.09017	0.00230
0.040	0.00020	0.00011	0.240	0.00722	0.00062	0.440	0.02450	0.00114	0.640	0.05258	0.00170	0.840	0.09247	0.00233
0.050	0.00031	0.00014	0.250	0.00784	0.00064	0.450	0.02564	0.00117	0.650	0.05428	0.00173	0.850	0.09480	0.00236
0.060	0.00045	0.00016	0.260	0.00848	0.00067	0.460	0.02681	0.00119	0.660	0.05601	0.00177	0.860	0.09716	0.00241
0.070	0.00061	0.00019	0.270	0.00915	0.00069	0.470	0.02800	0.00122	0.670	0.05778	0.00179	0.870	0.09957	0.00243
0.080	0.00080	0.00021	0.280	0.00984	0.00072	0.480	0.02922	0.00125	0.680	0.05957	0.00182	0.880	0.10200	0.00246
0.090	0.00101	0.00024	0.290	0.01056	0.00075	0.490	0.03047	0.00128	0.690	0.06139	0.00185	0.890	0.10446	0.00251
0.100	0.00125	0.00026	0.300	0.01131	0.00077	0.500	0.03175	0.00131	0.700	0.06324	0.00189	0.900	0.10697	0.00253
0.110	0.00151	0.00029	0.310	0.01208	0.00080	0.510	0.03306	0.00132	0.710	0.06513	0.00191	0.910	0.10950	0.00258
0.120	0.00180	0.00031	0.320	0.01288	0.00082	0.520	0.03438	0.00136	0.720	0.06704	0.00194	0.920	0.11206	0.00260
0.130	0.00211	0.00034	0.330	0.01370	0.00085	0.530	0.03574	0.00139	0.730	0.06898	0.00198	0.930	0.11468	0.00265
0.140	0.00245	0.00037	0.340	0.01455	0.00088	0.540	0.03713	0.00142	0.740	0.07096	0.00201	0.940	0.11733	0.00267
0.150	0.00282	0.00038	0.350	0.01543	0.00090	0.550	0.03855	0.00145	0.750	0.07297	0.00204	0.950	0.12000	0.00273
0.160	0.00320	0.00041	0.360	0.01633	0.00092	0.560	0.04000	0.00147	0.760	0.07501	0.00207	0.960	0.12273	0.00275
0.170	0.00361	0.00044	0.370	0.01725	0.00096	0.570	0.04147	0.00150	0.770	0.07708	0.00210	0.970	0.12548	0.00279
0.180	0.00405	0.00046	0.380	0.01821	0.00098	0.580	0.04297	0.00153	0.780	0.07918	0.00213	0.980	0.12827	0.00283
0.190	0.00451	0.00049	0.390	0.01919	0.00101	0.590	0.04450	0.00156	0.790	0.08131	0.00217	0.990	0.13110	0.00287
0.200	0.00500		0.400	0.02020		0.600	0.04606		0.800	0.08348		1.000	0.13397	

TABLE XII—Continued

Table for Computing Edge Thickness

DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.	DIA/R	a/R	DIF.
1.000	0.13397	0.00291	1.200	0.20000	0.00378	1.400	0.28586	0.00494	1.600	0.40000	0.00672	1.800	0.56411	0.01048
1.010	0.13688	0.00294	1.210	0.20377	0.00383	1.410	0.29079	0.00501	1.610	0.40672	0.00685	1.810	0.57459	0.01080
1.020	0.13982	0.00299	1.220	0.20760	0.00387	1.420	0.29580	0.00508	1.620	0.41357	0.00697	1.820	0.58539	0.01115
1.030	0.14281	0.00302	1.230	0.21147	0.00392	1.430	0.30088	0.00515	1.630	0.42054	0.00710	1.830	0.59655	0.01154
1.040	0.14583	0.00306	1.240	0.21540	0.00398	1.440	0.30603	0.00522	1.640	0.42764	0.00723	1.840	0.60808	0.01195
1.050	0.14890	0.00311	1.250	0.21938	0.00403	1.450	0.31125	0.00530	1.650	0.43487	0.00737	1.850	0.62003	0.01241
1.060	0.15200	0.00314	1.260	0.22341	0.00408	1.460	0.31655	0.00538	1.660	0.44224	0.00751	1.860	0.63244	0.01291
1.070	0.15515	0.00319	1.270	0.22749	0.00414	1.470	0.32193	0.00546	1.670	0.44975	0.00766	1.870	0.64535	0.01347
1.080	0.15833	0.00323	1.280	0.23162	0.00419	1.480	0.32739	0.00554	1.680	0.45741	0.00782	1.880	0.65883	0.01410
1.090	0.16156	0.00327	1.290	0.23581	0.00425	1.490	0.33293	0.00563	1.690	0.46523	0.00798	1.890	0.67293	0.01482
1.100	0.16483	0.00332	1.300	0.24006	0.00431	1.500	0.33856	0.00572	1.700	0.47322	0.00815	1.900	0.68775	0.01564
1.110	0.16815	0.00336	1.310	0.24437	0.00436	1.510	0.34428	0.00580	1.710	0.48137	0.00834	1.910	0.70339	0.01661
1.120	0.17151	0.00340	1.320	0.24873	0.00442	1.520	0.35008	0.00589	1.720	0.48970	0.00852	1.920	0.72000	0.01775
1.130	0.17491	0.00345	1.330	0.25316	0.00448	1.530	0.35597	0.00599	1.730	0.49823	0.00872	1.930	0.73775	0.01915
1.140	0.17836	0.00349	1.340	0.25764	0.00454	1.540	0.36195	0.00609	1.740	0.50695	0.00893	1.940	0.75689	0.02090
1.150	0.18185	0.00354	1.350	0.26218	0.00461	1.550	0.36804	0.00618	1.750	0.51588	0.00915	1.950	0.77780	0.02321
1.160	0.18538	0.00359	1.360	0.26679	0.00467	1.560	0.37422	0.00629	1.760	0.52503	0.00938	1.960	0.80100	0.02644
1.170	0.18897	0.00363	1.370	0.27146	0.00473	1.570	0.38050	0.00639	1.770	0.53441	0.00963	1.970	0.82745	0.03149
1.180	0.19260	0.00368	1.380	0.27619	0.00480	1.580	0.38689	0.00650	1.780	0.54404	0.00990	1.980	0.85893	0.04119
1.190	0.19627	0.00373	1.390	0.28099	0.00487	1.590	0.39339	0.00661	1.790	0.55393	0.01018	1.990	0.90013	0.09987
1.200	0.20000		1.400	0.28586		1.600	0.40000		1.800	0.56411		2.000	1.00000	

Example 45:

The known constants of the positive meniscus lens shown in diagram (e) are: $R_1 = 1.00$, $R_2 = 2.00$, DIA. $= 1.00$, and E.T. $= 0.10$. What is the axial thickness of the lens?

DIA/$R_1 = 1.00$ and DIA/$R_2 = 0.50$. From table XII we find,

$a_1/R_1 = 0.13397$, $a_2/R_2 = 0.03175$, $a_1 = 0.13397$, and

$a_2 = 0.06350$, hence, from equation (377)

$$d = a_1 - a_2 + \text{E.T.} = 0.1705 \text{ inches}$$

Example 46:

The negative meniscus lens shown in diagram (f) has the following characteristics:

$R_1 = 1.00$, $R_2 = 2.00$, DIA. $= 1.00$, $d = 0.100$. What is the edge thickness of the lens?

DIA/$R_1 = 1.00$ and DIA/$R_2 = 0.50$. From table XII we have, $a_1/R_1 = 0.13397$ and $a_2/R_2 = 0.03175$, hence, $a_1 = 0.13397$ and $a_2 = 0.0635$. Therefore, applying equation (378)

$$\text{E.T.} = a_1 - a_2 + d = 0.1705 \text{ inches}$$

Example 47:

The following constants are known for the plane-convex lens shown in diagram (g).

$R_1 = \pm\infty$, $R_2 = 1.00$, and $d = 0.25$. What will be the diameter of the lens if it is ground to knife-edge?

Here E.T. $= 0$ and $a_2 = d = 0.25$. Substituting these values into equation (379) we find,

$$\text{Diameter} = 2r = 2\sqrt{d(2R_2 - d)} = 2\sqrt{0.4375} = 1.3228 \text{ inches}$$

Example 48:

The positive meniscus lens shown in diagram (h) is ground to knife-edge. What is its diameter when $R_1 = 0.250$, $R_2 = 0.825$, and $d = 0.175$?

Here $d = a_1 - a_2$ and, substituting these values into equation (380) we find,

$$\text{Diameter} = \frac{\sqrt{(2R_1 - d)d\,\{4(d - R_1 + R_2)R_2 - (2R_1 - d)d\}}}{d - R_1 + R_2}$$

$$= \frac{\sqrt{0.056875 \times 2.418125}}{0.75} = \frac{0.370838}{0.75} = 0.4944 \text{ inches}$$

Example 49:

In diagram (i) is shown a bi-convex lens ground to knife-edge. What is its diameter of the lens when the radii are 1.382 and 0.334, respectively, and the axial thickness is 0.247?

Here $d = a_1 + a_2$ and, substituting these values into equation (381), we find,

$$\text{Diameter} = \frac{\sqrt{(2R_1 - d)d\,\{4(R_1 + R_2 - d)R_2 - (2R_1 - d)d\}}}{R_1 + R_2 - d} =$$

$$\frac{\sqrt{0.621699 \times 1.340885}}{1.469} = \frac{0.9130317}{1.469} = 0.6215 \text{ inches}$$

76. Volume and Weight of a Lens

When the volume and the weight of a lens are to be determined we assume the lens to be divided into three parts, namely, the segment S_1 formed by the first spherical surface, the segment S_2 formed by the second spherical surface, and the cylinder C included by the two segments, as illustrated in figure 99. The segments are shaded in the diagrams. The height of these segments are designated by the letters a_1 and a_2 and are given in table XII. a_3 is the height of the cylinder and is, in all cases, equal to the edge thickness of the lens. The volume of the lens is denoted by the letter V, the volumes of the two segments by V_1 and V_2 respectively, and the volume of the cylinder by V_3. After computing the volume V we determine the weight W of the lens.

Again, all radii will be considered positive.

W_u is the weight in pounds for one cubic inch of the media.

In table XIII are tabulated the W_u values for the glasses commonly used in the manufacture of optical systems for fire control instruments.

Formulae

$$V_1 = \pi a_1^2 \left\{R_1 - \frac{a_1}{3}\right\} = \frac{a_1^2(3R_1 - a_1)}{0.9549297} = \frac{a_1(3r^2 + a_1^2)}{1.90956}$$

$$V_2 = \pi a_2^2 \left\{R_2 - \frac{a_2}{3}\right\} = \frac{a_2^2(3R_2 - a_2)}{0.9549297} = \frac{a_2(3r^2 + a_2^2)}{1.90956} \qquad (382)$$

$$V_3 = \pi r^2(ET) = \frac{a_3 r^2}{0.3183044}$$

For the different shape lenses the volume V is computed for each case. The formulae for computing the value of $a_3 = $ E.T. have already been given; however, for convenience they are given again.

Case 1. (fig. 99(a))

Bi-convex lens: $a_3 = d - a_1 - a_2$ $\quad V = V_1 + V_2 + V_3$ (383)

Case 2. (fig. 99(b))

Plane-convex lens: $a_3 = d - a_2$ $\quad V = V_2 + V_3$ (384)

Case 3. (fig. 99(c))

Plane-concave lens: $a_3 = d + a_2$ $\quad V = V_3 - V_2$ (385)

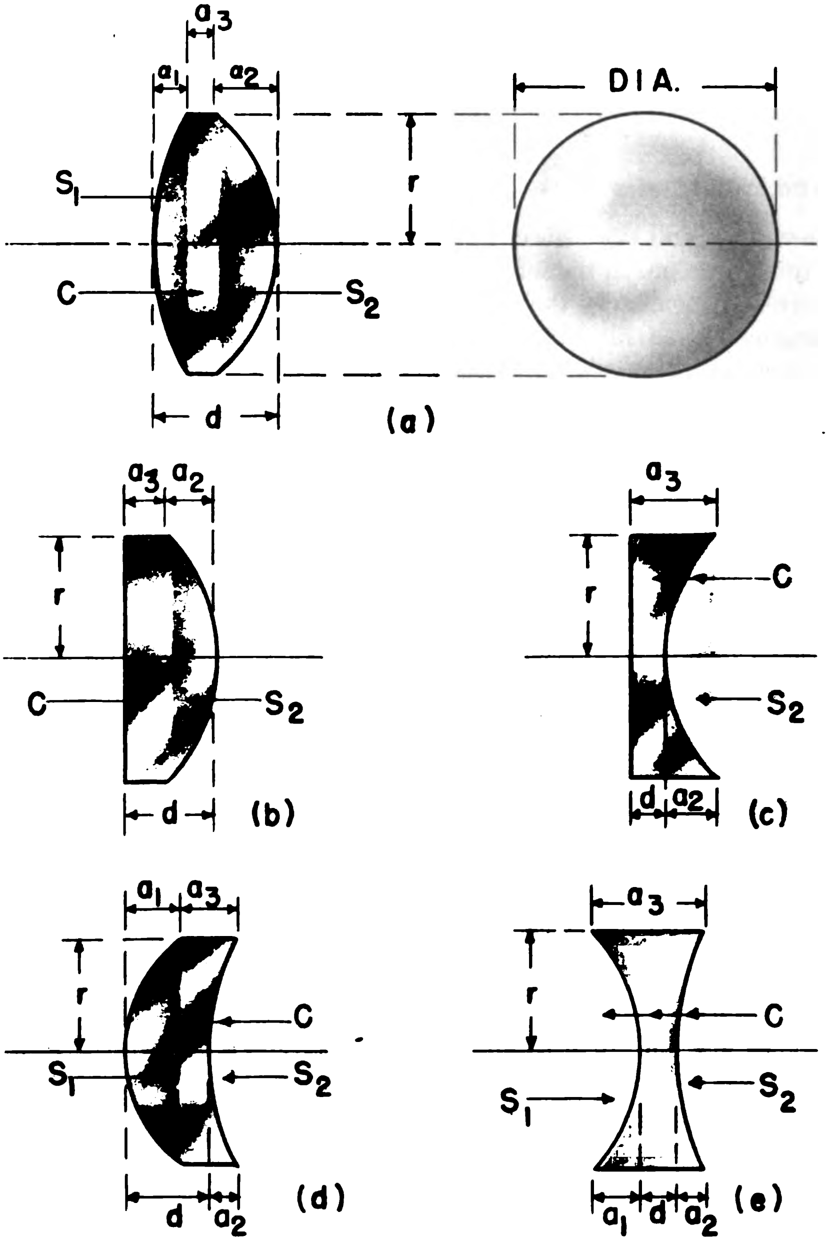


Figure 99. The volume and the weight of a lens.

TABLE XIII

MEDIA		Sp.G.	W_u (in pounds)	W_u (in ounces)
WATER		1.00	0.036125	0.578
Borosilicate Crown	BSC–1	2.48	0.0896	1.434
Borosilicate Crown	BSC–2	2.53	0.0914	1.462
Crown	C–1	2.53	0.0914	1.462
Crown	C–2	2.11	0.0762	1.219
Light Barium Crown	LBC–1	2.85	0.1030	1.648
Light Barium Crown	LBC–2	3.21	0.1160	1.856
Light Barium Crown	LBC–3	3.20	0.1156	1.850
Dense Barium Crown	DBC–1	3.58	0.1293	2.069
Dense Barium Crown	DBC–2	3.66	0.1322	2.115
Dense Barium Crown	DBC–3	3.57	0.1290	2.064
Crown-Flint	CF–1	2.73	0.0986	1.578
Light Barium Flint	LBF–1	3.33	0.1203	1.925
Extra Light Flint	ELF–1	3.05	0.1102	1.763
Extra Light Flint	ELF–2	2.95	0.1066	1.706
Barium Flint	BF–1	3.31	0.1196	1.914
Barium Flint	BF–2	3.52	0.1272	2.035
Light Flint	LF–1	3.19	0.1152	1.843
Light Flint	LF–2	3.27	0.1181	1.890
Light Flint	LF–3	3.21	0.1160	1.856
Dense Flint	DF–1	3.49	0.1261	2.018
Dense Flint	DF–2	3.64	0.1315	2.104
Dense Flint	DF–3	3.67	0.1326	2.122
Dense Barium Flint	DBF–1	3.61	0.1304	2.086
Extra Dense Flint	EDF–1	3.91	0.1412	2.259
Extra Dense Flint	EDF–2	4.24	0.1532	2.451
Extra Dense Flint	EDF–3	4.51	0.1629	2.606
	EK–11	4.10	0.1481	2.370
	EK–31	4.50	0.1626	2.602
	EK–32	4.48	0.1618	2.590
	EK–33	4.65	0.1680	2.688

Case 4. (fig. 99(d))

Positive meniscus lens: $a_3 = d - a_1 + a_2 \qquad V = V_1 + V_3 - V_2$ (386)

Case 5. (fig. 99(e))

Bi-concave lens: $a_3 = d + a_1 + a_2 \qquad V = V_3 - V_1 - V_2$ (387)

Case 6. (fig. 99(f))

Negative meniscus lens: $a_3 = d + a_1 - a_2 \qquad V = V_3 - V_1 + V_2$ (388)

With these constants determined we are able to compute the weight W of the lens with the following equation:

$$W = VW_u \tag{389}$$

Where the value of W_u is taken from table XIII.

The value of V can be determined very easily from table XIV.

Example 50:

The bi-convex lens shown in figure 99(a) is made of the Light Barium Crown glass LBC–2. Its radii are 1 inch and 0.75 inch, respectively. The diameter is 1 inch and the axial thickness is 0.50 inch. What are the volume and the weight (in ounces) of this lens?

We have, $R_1 = 1$ DIA. $= 1.00$

From table XIII: $W_u = 1.856$ $R_2 = 0.75$ $d = 0.50$

From table XII: $Dia/R_1 = 1.000$ $a_1/R_1 = 0.13397$ and $a_1 = 0.13397$

From table XII: $Dia/R_2 = 1.3333$ $a_2/R_2 = 0.25465$ and $a_2 = 0.19099$

From equation (383): $a_3 = d - a_1 - a_2 = 0.17504$

From table XIV: $Dia/R_1 = 1.000$ $V_1/R_1^3 = 0.05387$ and $V_1 = 0.05387$

From table XIV: $Dia/R_2 = 1.33335$ $V_2/R_2^3 = 0.18638$ and $V_2 = 0.07863$

From equation (382): $V_3 = 0.13748$

From equation (383): $V = V_1 + V_2 + V_3 = 0.26997$ cubic inches

From equation (389): $W = 0.501$ ounces

Example 51:

The convex surface of the plane-convex lens shown in figure 99(b) is 1 inch, and the diameter is also 1 inch, and the axial thickness is equal to 0.25 inch. What are the volume and the weight (in ounces) of this lens?

We have, $R_1 = \pm \infty$ DIA. $= 1.00$

From table XIII: $W_u = 1.434$ $R_2 = 1.00$ $d = 0.25$

From table XII: $Dia/R_2 = 1.000$ $a_2/R_2 = 0.13397$ $a_2 = 0.13397$

From equation (384): $a_3 = d - a_2 = 0.11603$

From table XIV: $Dia/R_2 = 1.000$ $V_2/R_2^3 = 0.05387$ $V_2 = 0.05387$

From equation (382): $V_3 = 0.09113$

From equation (384): $V = 0.145$ cubic in.

From equation (389): $W = 0.208$ ounces

Example 52:

The second surface of the plane-concave lens shown in figure 99(c) has a radius of 1 inch. The lens, which is made of the Dense Flint glass DF–3, has a diameter of 0.75 inch and an axial thickness of 0.10 inch. What are the volume and the weight (in ounces) of the lens?

We have, $R_1 = \pm \infty$ Dia. $= 0.75$

From table XIII: $W_u = 2.122$ $R_2 = 1.00$ $d = 0.10$

TABLE XIV

Table Used for Computing the Volume of a Spherical Segment

DIA/R	V/R^3	DIF.	DIA/R	V/R^3	DIF.	DIA/R	V/R^3	DIF.	DIA/R	V/R^3	DIF.	DIA/R	V/R^3	DIF.
0.000	0.00000	0.00000	0.200	0.00008	0.00002	0.400	0.00127	0.00013	0.600	0.00656	0.00046	0.800	0.02129	0.00111
0.010	0.00000	0.00000	0.210	0.00010	0.00002	0.410	0.00140	0.00015	0.610	0.00702	0.00048	0.810	0.02240	0.00117
0.020	0.00000	0.00000	0.220	0.00012	0.00002	0.420	0.00155	0.00015	0.620	0.00750	0.00050	0.820	0.02357	0.00121
0.030	0.00000	0.00000	0.230	0.00014	0.00002	0.430	0.00170	0.00017	0.630	0.00800	0.00053	0.830	0.02478	0.00126
0.040	0.00000	0.00000	0.240	0.00016	0.00003	0.440	0.00187	0.00018	0.640	0.00853	0.00056	0.840	0.02604	0.00130
0.050	0.00000	0.00000	0.250	0.00019	0.00003	0.450	0.00205	0.00019	0.650	0.00909	0.00058	0.850	0.02734	0.00136
0.060	0.00000	0.00000	0.260	0.00022	0.00004	0.460	0.00224	0.00020	0.660	0.00967	0.00062	0.860	0.02870	0.00141
0.070	0.00000	0.00000	0.270	0.00026	0.00004	0.470	0.00244	0.00022	0.670	0.01029	0.00064	0.870	0.03011	0.00146
0.080	0.00000	0.00000	0.280	0.00030	0.00005	0.480	0.00266	0.00023	0.680	0.01093	0.00067	0.880	0.03157	0.00152
0.090	0.00000	0.00000	0.290	0.00035	0.00005	0.490	0.00289	0.00024	0.690	0.01160	0.00070	0.890	0.03309	0.00158
0.100	0.00000	0.00001	0.300	0.00040	0.00006	0.500	0.00313	0.00027	0.700	0.01230	0.00074	0.900	0.03467	0.00163
0.110	0.00001	0.00000	0.310	0.00046	0.00006	0.510	0.00340	0.00027	0.710	0.01304	0.00077	0.910	0.03630	0.00168
0.120	0.00001	0.00000	0.320	0.00052	0.00007	0.520	0.00367	0.00030	0.720	0.01381	0.00080	0.920	0.03798	0.00176
0.130	0.00001	0.00001	0.330	0.00059	0.00007	0.530	0.00397	0.00031	0.730	0.01461	0.00084	0.930	0.03974	0.00182
0.140	0.00002	0.00000	0.340	0.00066	0.00008	0.540	0.00428	0.00033	0.740	0.01545	0.00087	0.940	0.04156	0.00187
0.150	0.00002	0.00001	0.350	0.00074	0.00009	0.550	0.00461	0.00035	0.750	0.01632	0.00092	0.950	0.04343	0.00196
0.160	0.00003	0.00001	0.360	0.00083	0.00010	0.560	0.00496	0.00037	0.760	0.01724	0.00095	0.960	0.04539	0.00201
0.170	0.00004	0.00001	0.370	0.00093	0.00011	0.570	0.00533	0.00039	0.770	0.01819	0.00099	0.970	0.04740	0.00208
0.180	0.00005	0.00001	0.380	0.00104	0.00011	0.580	0.00572	0.00041	0.780	0.01918	0.00103	0.980	0.04948	0.00216
0.190	0.00006	0.00002	0.390	0.00115	0.00012	0.590	0.00613	0.00043	0.790	0.02021	0.00108	0.990	0.05164	0.00223
0.200	0.00008		0.400	0.00127		0.600	0.00656		0.800	0.02129		1.000	0.05387	

TABLE XIV—Continued

Table Used for Computing the Volume of a Spherical Segment

DIA/R	V/R³	DIF.	DIA/R	V/R³	DIF.	DIA/R	V/R³	DIF.	DIA/R	V/R³	DIF.	DIA/R	V/R³	DIF.
1.000	0.05387	0.00231	1.200	0.11729	0.00430	1.400	0.23225	0.00765	1.600	0.43563	0.01360	1.800	0.81173	0.02682
1.010	0.05618	0.00238	1.210	0.12159	0.00444	1.410	0.23990	0.00788	1.610	0.44923	0.01403	1.810	0.83855	0.02795
1.020	0.05856	0.00246	1.220	0.12603	0.00456	1.420	0.24778	0.00810	1.620	0.46326	0.01446	1.820	0.86650	0.02919
1.030	0.06102	0.00254	1.230	0.13059	0.00470	1.430	0.25588	0.00833	1.630	0.47772	0.01490	1.830	0.89569	0.03049
1.040	0.06356	0.00263	1.240	0.13529	0.00485	1.440	0.26421	0.00856	1.640	0.49262	0.01537	1.840	0.92618	0.03195
1.050	0.06619	0.00272	1.250	0.14014	0.00498	1.450	0.27277	0.00882	1.650	0.50799	0.01585	1.850	0.95813	0.03354
1.060	0.06891	0.00280	1.260	0.14512	0.00513	1.460	0.28159	0.00906	1.660	0.52384	0.01636	1.860	0.99167	0.03527
1.070	0.07171	0.00289	1.270	0.15025	0.00528	1.470	0.29065	0.00933	1.670	0.54020	0.01688	1.870	1.02694	0.03722
1.080	0.07460	0.00299	1.280	0.15553	0.00544	1.480	0.29998	0.00960	1.680	0.55708	0.01744	1.880	1.06416	0.03935
1.090	0.07759	0.00308	1.290	0.16097	0.00559	1.490	0.30958	0.00988	1.690	0.57452	0.01802	1.890	1.10351	0.04180
1.100	0.08067	0.00318	1.300	0.16656	0.00576	1.500	0.31946	0.01017	1.700	0.59254	0.01862	1.900	1.14531	0.04459
1.110	0.08385	0.00328	1.310	0.17232	0.00593	1.510	0.32963	0.01046	1.710	0.61116	0.01924	1.910	1.18990	0.04784
1.120	0.08713	0.00338	1.320	0.17825	0.00610	1.520	0.34009	0.01076	1.720	0.63040	0.01993	1.920	1.23774	0.05166
1.130	0.09051	0.00349	1.330	0.18435	0.00627	1.530	0.35085	0.01107	1.730	0.65033	0.02062	1.930	1.28940	0.05629
1.140	0.09400	0.00359	1.340	0.19062	0.00646	1.540	0.36192	0.01141	1.740	0.67095	0.02135	1.940	1.34569	0.06213
1.150	0.09759	0.00370	1.350	0.19708	0.00664	1.550	0.37333	0.01174	1.750	0.69230	0.02214	1.950	1.40782	0.06965
1.160	0.10129	0.00383	1.360	0.20372	0.00684	1.560	0.38507	0.01208	1.760	0.71444	0.02295	1.960	1.47747	0.08022
1.170	0.10512	0.00393	1.370	0.21056	0.00702	1.570	0.39715	0.01245	1.770	0.73739	0.02383	1.970	1.55769	0.09646
1.180	0.10905	0.00405	1.380	0.21758	0.00723	1.580	0.40960	0.01283	1.780	0.76122	0.02476	1.980	1.65415	0.12754
1.190	0.11310	0.00419	1.390	0.22481	0.00744	1.590	0.42243	0.01320	1.790	0.78598	0.02575	1.990	1.78169	0.31270
1.200	0.11729		1.400	0.23225		1.600	0.43563		1.800	0.81173		2.000	2.09439	

From table XII: $DIA/R_2 = 0.75$ $a_2/R_2 = 0.07297$ $a_2 = 0.07297$

From equation (385): $a_3 = 0.17297$

From table XIV: $DIA/R_2 = 0.75$ $V_2/R_2^3 = 0.01632$ $V_2 = 0.01632$

From equation (382): $V_3 = 0.0764158$

From equation (385): $V = 0.06010$ cu. in.

From equation (389): $W = 0.128$ ounces

Example 53:

The positive meniscus lens illustrated in figure 99(d) is made of the Light Barium Crown glass LBC–2, and has a diameter of 1 inch. The radii, 1 inch and 2 inches, respectively, while the axial thickness is 0.25 inch. What are the volume and the weight (in ounces) of this lens?

We have, $R_1 = 1.00$ DIA. $= 1.00$

From table XIII: $W_u = 1.856$ $R_2 = 2.00$ $d = 0.25$

From table XII $Dia/R_1 = 1.000$ $a_1/R_1 = 0.13397$ $a_1 = 0.13397$

$Dia/R_2 = 0.500$ $a_2/R_2 = 0.03175$ $a_2 = 0.06350$

From equation (386): $a_3 = 0.17953$

From table XIV: $Dia/R_1 = 1.000$ $V_1/R_1^3 = 0.05387$ $V_1 = 0.05387$

$Dia/R_2 = 0.500$ $V_2/R_2^3 = 0.00313$ $V_2 = 0.02504$

From equation (382): $V_3 = 0.14100$

From equation (386): $V = 0.16983$ cu. in.

From equation (389): $W = 0.315$ ounces

Example 54:

The bi-concave lens illustrated in figure 99(e) is one inch in diameter and its radii are 2 inches and 1 inch, respectively. The lens, whose axial thickness is equal to 0.10 inch, is made of the Extra Dense Flint glass EDF–3. What are the volume and the weight (in ounces) of the lens?

We have, $R_1 = 2.00$ Dia. $= 1.00$

From table XIII: $W_u = 2.606$ $R_2 = 1.00$ $d = 0.10$

From tables XII and IX, and equations (387), (382), and (389), we find,

From tables XII and XIV, and equations (387), (382), and (389), we find,

$$W = 0.403 \text{ ounces}$$

Example 55:

A plane-parallel disk, 3 inches in diameter, has a thickness of 1 inch. What is the weight if the disk is made of (a) BSC–1 and (b) SFS–1?

We have, $a = 1$ inch and $2r = 3$ inches. From table XIII,

$W_u = 1.434$ (case a) and $W_u = 3.485$ (case b)

The volume of the plate is computed with equation (382)

$$V = \frac{ar^2}{0.3183099} = \frac{2.25}{0.3183099} = 7.06858 \text{ cubic inches}$$

Applying equation (389) we find,

Case (a) $W = VW_u = 7.06858 \times 1.434 = 10.14$ ounces

Case (b) $W = VW_u = 7.06858 \times 3.485 = 24.63$ ounces

77. Fundamental Formulae for Thick Lens Systems

In discussing thick lens systems we make the following assumptions:

a. The lens system is in air, which is generally the case in fire control instruments.

b. The radii R_v are given.

c. The axial thickness of the lenses are known.

d. The air-spaces d_v are given.

e. The refractive indices n_v of the various media are known.

We can always trace two zero-rays ($s_1 = \infty$), in opposite directions, through the system and thus determine the constants of the combination. However, Ordnance Corps drawings must show the constants of each unit of the combination; therefore, their values must be computed. In order to generalize the formulae to be used in computing the focal lengths and the positions of the Gauss points and planes, let us imagine that each lens, whether cemented to another lens or not, is surrounded by air. In other words, we assume an air lens of zero thickness between the cemented lenses. Therefore, the focal lengths of the collective lens and the dispersive lens of the cemented doublet, illustrated in the top diagram of figure 100, are designated by f_1 and f_3, respectively, and that of the air lens between them by f_2. The axial thicknesses d carry the same subscript. The focal length of an air lens is $f = \infty$ and $d = 0$ in all cases. The dotted vertical lines indicate the positions of the principal planes of the individual lenses of the system while the corresponding planes of the combination are indicated by full vertical lines. The two lenses of the doublet are separated by the optical distance D_1 which is the distance between the second principal plane of the first lens and the first principal plane of the second lens as shown in the diagrams. With these definitions fixed the formulae given in the following can be applied to any combination of thick lenses.

$$a_v = n_v(R_{v+1} - R_v) + (n_v - 1)d_v$$

$$b_v = (n_v - 1)a_v \qquad c_v = n_v R_v R_{v+1} \qquad f_v = \frac{c_v}{b_v} \tag{390}$$

$$H_v = -\frac{d_v R_v}{a_v} \qquad H_v' = -\frac{d_v R_{v+1}}{a_v}$$

$$D_v = d_{2v} - H'_{2v-1} + H_{2v+1}$$

Note: For a cemented surface, $n = 1$ and $d = 0$, hence, $f = H = H' = 0$

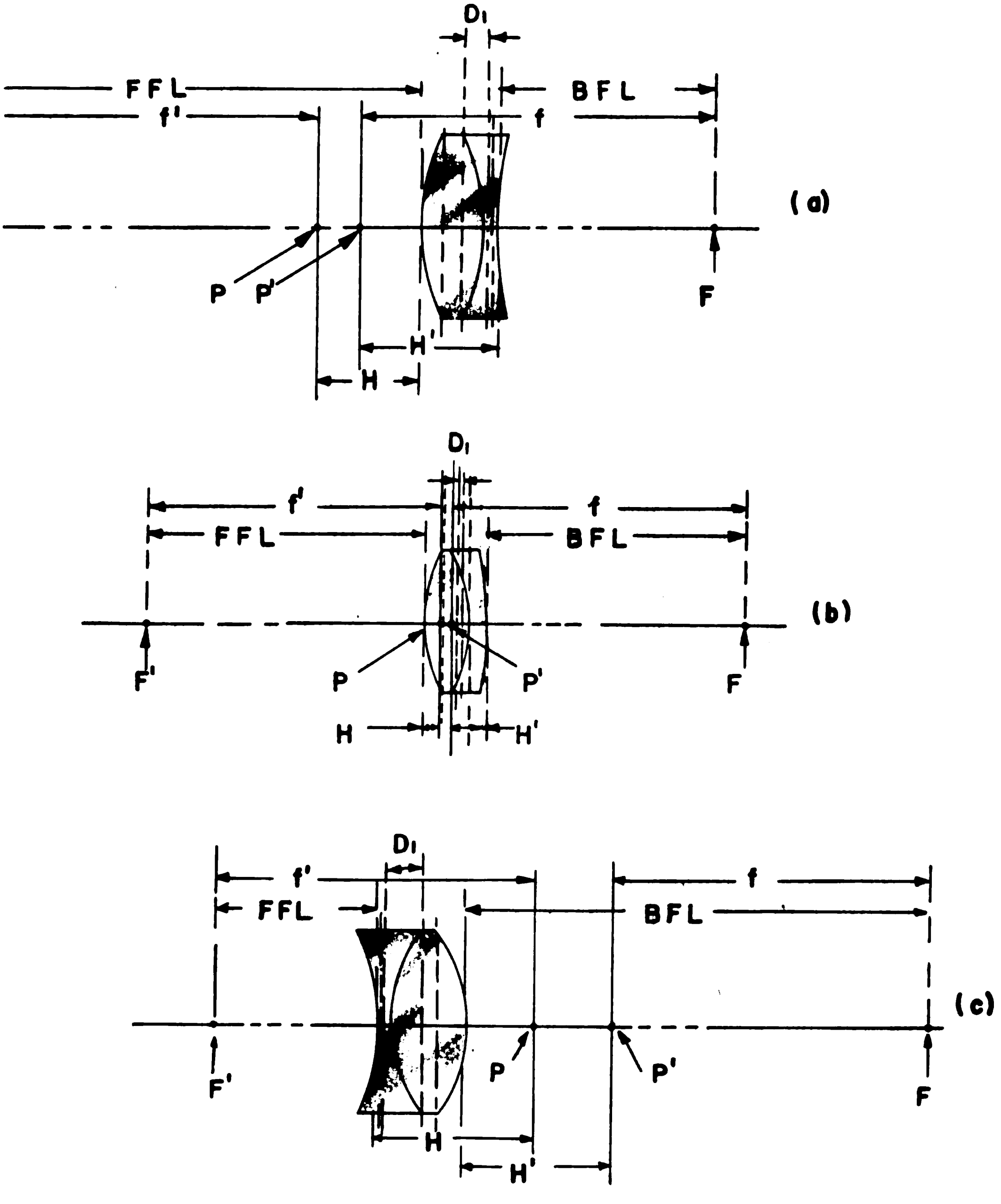


Figure 100. The thick cemented doublet.

Special cases

a. The first surface of the vth lens is plane, that is, $R_v = \pm \infty$ then

$$a_v = b_v = c_v = \pm \infty \qquad H_v' = \pm 0$$

$$f_v = -\frac{R_{v+1}}{n_v - 1} \qquad H_v = +\frac{d_v}{n_v} \tag{391}$$

b. The second surface of the vth lens is plane, that is $R_{v+1} = \pm \infty$ then

$$a_v = b_v = c_v = \pm \infty \qquad H_v = \pm 0$$

$$f_v = +\frac{R_v}{n_v - 1} \qquad H_v' = -\frac{d_v}{n_v} \tag{392}$$

78. Two Thick Lenses Separated by Distance D

In the diagrams of figures 100 and 101 are shown two thick lenses separated by the distance D_1. The cemented doublets illustrated in figure 100 were selected to enhance the influence of the shape of the doublet upon the positions of the principal points P and P′. There are cases where the second principal plane of the first lens lies behind the first principal plane of the second lens, and in this case D_1 will have a negative value.

When computing the required optical data for any optical system, proceed as follows:

a. Write down the known constants of the system as shown at the head of table XV. Remember that cemented surfaces are counted as two, that the thickness of the air-lens between two cemented surfaces is $d = 0$, and that the refractive index of the medium is $n = 1$.

b. Substitute these values into equation (390).

c. Finally substitute the values into the following equations, in which the lenses are counted in the usual manner:

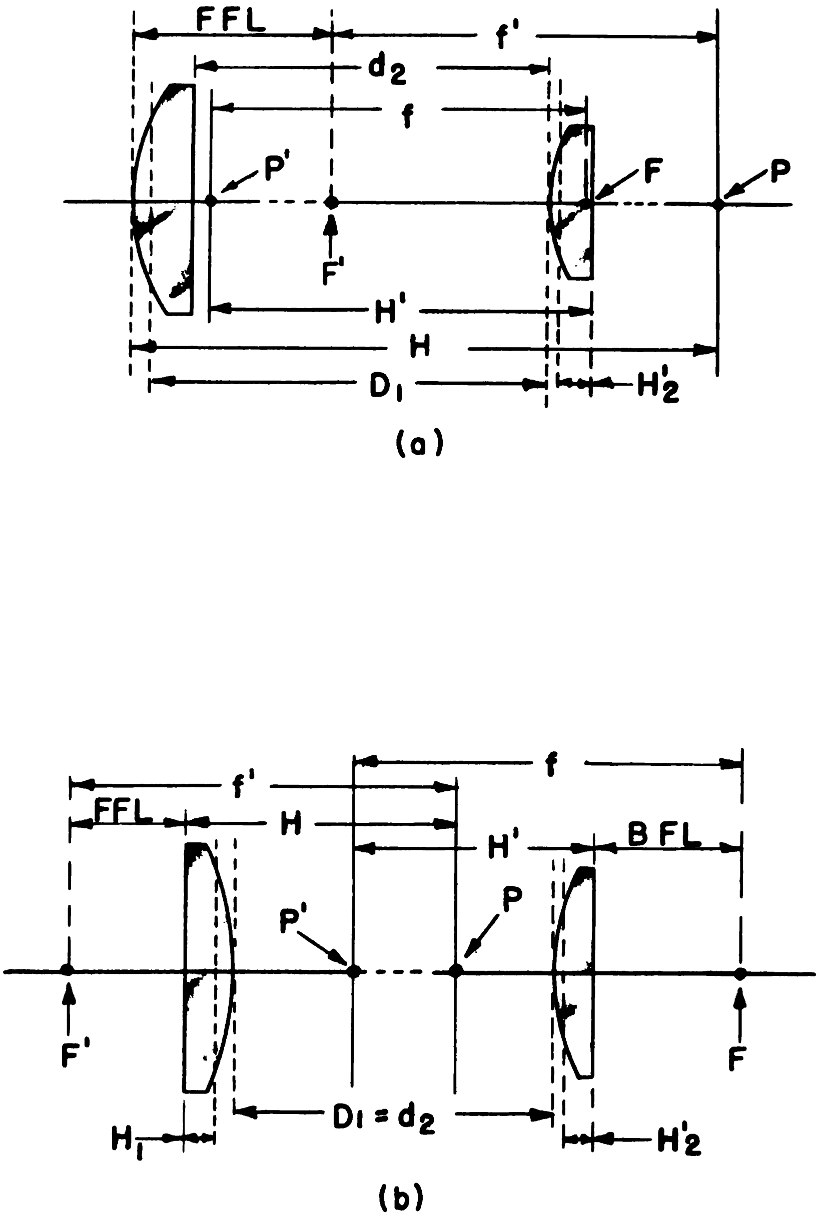
$$f = -f' = \frac{f_1 f_2}{f_1 + f_2 - D_1} \tag{393}$$

$$H = H_1 + \frac{D_1 f_1}{f_1 + f_2 - D_1} \tag{394}$$

$$H' = H_2' - \frac{D_1 f_2}{f_1 + f_2 - D_1} \tag{395}$$

$$\text{F.F.L.} = H_1 - \frac{(f_2 - D_1) f_1}{f_1 + f_2 - D_1} = H - f \tag{396}$$

$$\text{B.F.L.} = H_2' + \frac{(f_1 - D_1) f_2}{f_1 + f_2 - D_1} = H' + f \tag{397}$$



(a) The Huygenian eye piece.
(b) The Ramsden eye piece.

Figure 101.

In table XV are given the complete data for the optical systems shown in the diagrams of figures 100 and 101.

TABLE XV

INITIAL CONSTANTS

Example No. Figure No.	56 100(a)	57 100(b)	58 100(c)	59 101(a)	60 101(b)
$R_1 =$	+ 1.2011	+ 1.412	— 2.0015	+ 1.1363	± ∞
$R_2 =$	— 1.2011	— 0.870	+ 0.9088	± ∞	— 1.5495
$R_3 =$	— 1.2011	— 0.870	+ 0.9088	+ 0.7023	+ 1.3773
$R_4 =$	+ 4.7242	— 1.982	— 0.9088	± ∞	± ∞
$d_1 =$	0.4334	0.263	0.0937	0.30	0.25
$d_2 =$	0.0000	0.000	0.0000	1.8662	1.6667
$d_3 =$	0.0671	0.063	0.4686	0.20	0.20
$n_1 =$	1.5163	1.5170	1.6490	1.5170	1.5170
$n_2 =$	1.0000	1.0000	1.0000	1.0000	1.0000
$n_3 =$	1.7174	1.6490	1.5170	1.5170	1.5170

COMPUTED CONSTANTS

In the computed constants all values which we know to be either ± 0 or ± ∞ (such as H_2 or F_2) have been omitted. The conventional system of numbering glass elements consecutively is used rather than the extended system involving air lenses.

Example No. Figure No.	56 100(a)	57 100(b)	58 100(c)	59 101(a)	60 101(b)
$a_1 =$	— 3.4187	— 3.3258	+ 4.8599		
$a_2 =$	+ 10.2242	— 1.7928	— 2.5150		
$b_1 =$	— 1.7651	— 1.7195	+ 3.1541		
$b_2 =$	+ 7.3348	— 1.1635	— 1.3003		
$c_1 =$	— 2.1874	— 1.8635	— 2.9995		
$c_2 =$	— 9.7449	+ 2.8434	— 1.2529		
$f_1 =$	+ 1.2392	+ 1.0838	— 0.9510	+ 2.1979	+ 2.9971
$f_2 =$	— 1.3286	— 2.4438	+ 0.9635	+ 1.3584	+ 2.6640
$H_1 =$	+ 0.1523	+ 0.1117	+ 0.0386	± 0.0000	+ 0.1648
$H_1' =$	— 0.1523	— 0.0688	— 0.0175	— 0.1978	± 0.0000
$H_2 =$	+ 0.0079	— 0.0306	+ 0.1693	± 0.0000	± 0.0000
$H_2' =$	— 0.0310	— 0.0696	— 0.1693	— 0.1318	— 0.1318
$D_1 =$	0.1602	0.0382	0.1868	2.0640	1.6667
$f = -f' =$	+ 6.5962	+ 1.8943	+ 5.2570	+ 2.0007	+ 1.9989
$H =$	— 0.6431	+ 0.0821	+ 1.0578	+ 3.0399	+ 1.4154
$H' =$	— 0.8837	— 0.1364	+ 0.8633	— 2.0106	— 1.2434
F.F.L. =	— 7.2392	— 1.8122	— 4.1992	+ 1.0392	— 0.5835
B.F.L. =	+ 5.7124	+ 1.7579	+ 6.1203	— 0.0099	+ 0.7555

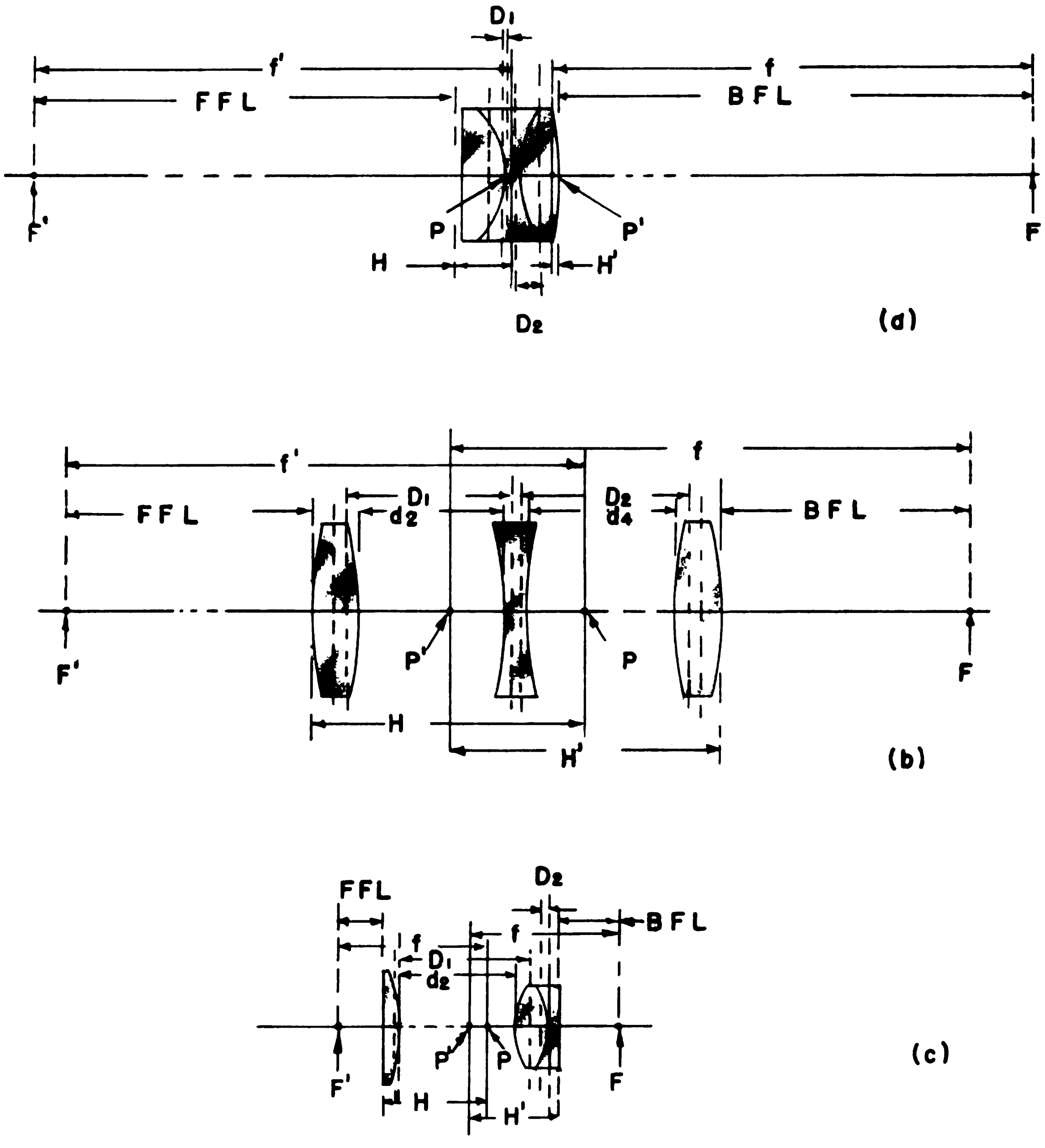


Figure 102. Three thick lenses separated by the distances D_1 and D_2.

79. Three Thick Lenses Separated by Distances D_1 and D_2

In figure 102 are shown three combinations, each consisting of three thick lenses separated by the distances D_1 and D_2. First, we substitute the known constants into equation (390) and then we apply the following set of equations, in which the lenses are counted in the usual manner. The following abbreviation will be used here:

$$J = (f_1 - D_1) f_2 + (f_1 + f_2 - D_1)(f_3 - D_2)$$

$$f = -f' = \frac{f_1 f_2 f_3}{J} \tag{398}$$

$$H = H_1 + \frac{\{f_2 D_2 + (f_2 + f_3 - D_2) D_1\} \; f_1}{J} \tag{399}$$

$$H' = H_3' - \frac{\{f_2 D_1 + (f_1 + f_2 - D_1) D_2\} \; f_3}{J} \tag{400}$$

$$\text{F.F.L.} = H_1 - \frac{\{(f_3 - D_2) f_2 - (f_2 + f_3 - D_2) D_1\} \; f_1}{J} = H - f \tag{401}$$

$$\text{B.F.L.} = H_3' + \frac{\{(f_1 - D_1) f_2 - (f_1 + f_2 - D_1) D_2\} \; f_3}{J} = H' + f \tag{402}$$

The known constants of the systems, as well as the results of the computations, are given in table XVI.

TABLE XVI

INITIAL CONSTANTS

Example Figure	61 102(a)	62 102(b)	63 102(c)
$R_1 =$	+ 27.1767	+ 4.00	± ∞
$R_2 =$	— 0.3035	— 3.00	— 0.9306
$R_3 =$	— 0.3035	— 3.00	+ 0.6689
$R_4 =$	+ 0.4657	+ 3.00	— 0.4839
$R_5 =$	+ 0.4657	+ 3.00	— 0.4839
$R_6 =$	— 1.6652	— 4.00	± ∞
$d_1 =$	0.160	0.30	0.1435
$d_2 =$	± 0.000	1.00	0.7923
$d_3 =$	0.035	0.15	0.2768
$d_4 =$	± 0.000	1.00	± 0.0000
$d_5 =$	0.120	0.30	0.0503
$n_1 =$	1.5163	1.5170	1.5170
$n_2 =$	1.0000	1.0000	1.0000
$n_3 =$	1.5473	1.6170	1.5725
$n_4 =$	1.0000	1.0000	1.0000
$n_5 =$	1.6123	1.5170	1.6170

TABLE XVI—Continued

COMPUTED CONSTANTS

Example Figure	61 102(a)	62 102(b)	63 102(c)
$f_1 =$	+ 0.583	+ 3.365	+ 1.8000
$f_2 =$	— 0.330	— 2.408	+ 0.5374
$f_3 =$	+ 0.607	+ 3.365	— 0.7843
$D_1 =$	+ 0.010	+ 1.132	+ 0.9042
$D_2 =$	+ 0.030	+ 1.132	+ 0.0810
$f = -f'$	2.367	4.727	1.0000
$H =$	+ 0.193	+ 1.820	+ 0.6947
$H' =$	— 0.010	— 1.820	— 0.6534
F.F.L. =	— 2.174	— 2.907	— 0.3054
B.F.L. =	+ 2.356	+ 2.907	+ 0.3466

In table XVI are given all the constants for each individual lens as well as for each combination. However, in the case of the Kellner eye piece shown in figure 102(c), the drawing must show the constants for the doublet (eye lens) as a unit. Therefore, we apply equations (393) to (397) but changing the subscripts to 23. The focal length of the doublet will now be f_{23}. We have,

$$f_{23} = -f_{23}' = \frac{f_2 f_3}{f_2 + f_3 - D_2} = +1.2857 \tag{403}$$

$$H_{23} = H_2 + \frac{D_2 f_2}{f_2 + f_3 - D_2} = -0.0208 \tag{404}$$

$$H_{23}' = H_3' - \frac{D_2 f_3}{f_2 + f_3 - D_2} = -0.2248 \tag{405}$$

$$(\text{F.F.L.})_{23} = H_{23} - f_{23} = -1.3065 \tag{406}$$

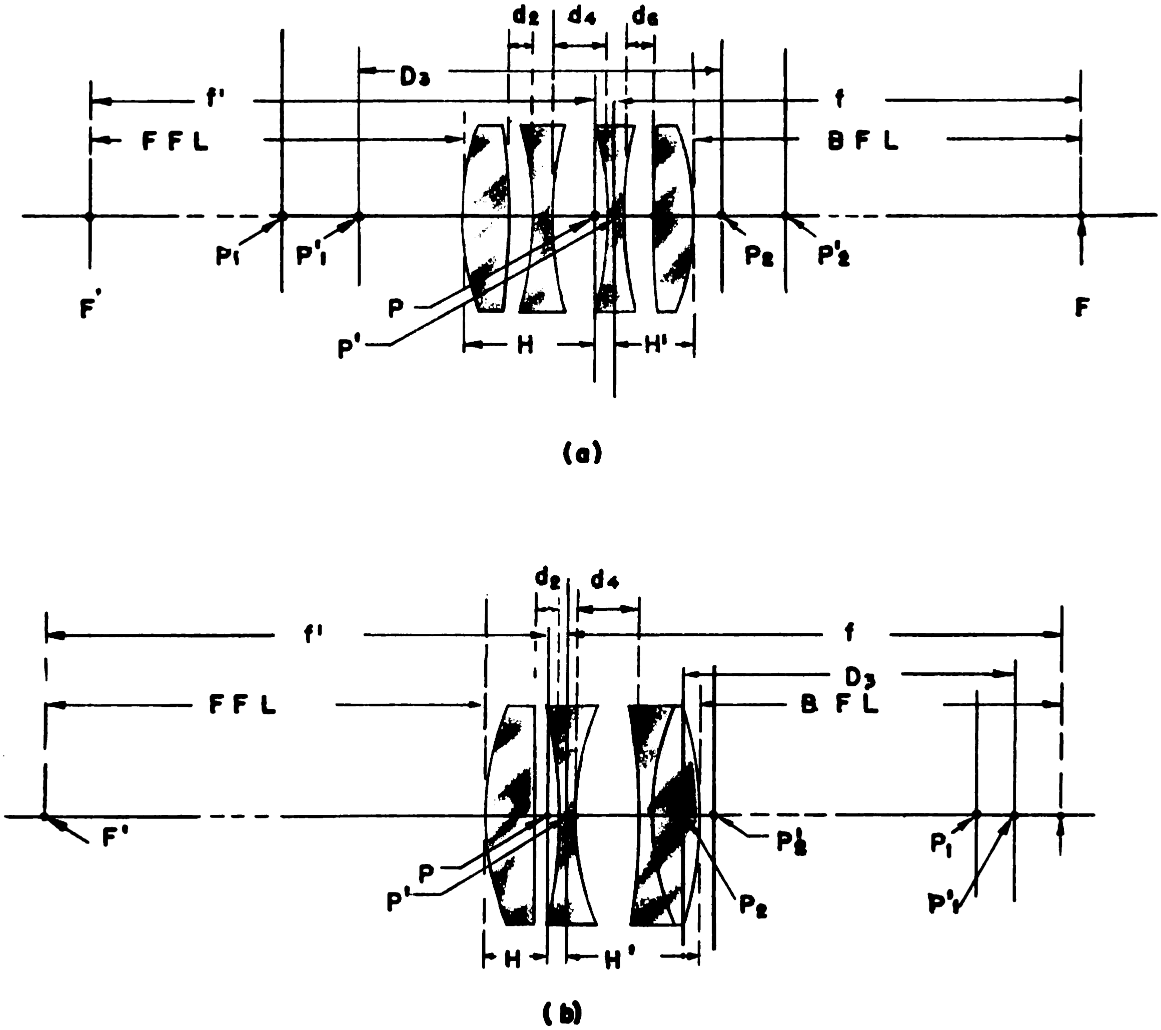
$$(\text{B.F.L.})_{23} = H_{23}' + f_{23} = +1.0609 \tag{407}$$

80. Combinations Consisting of Four or More Thick Lenses

Various optical systems, consisting of from four to six lenses are shown in **figures** 103, 104, and 105. The given constants for each combination are tabulated below.

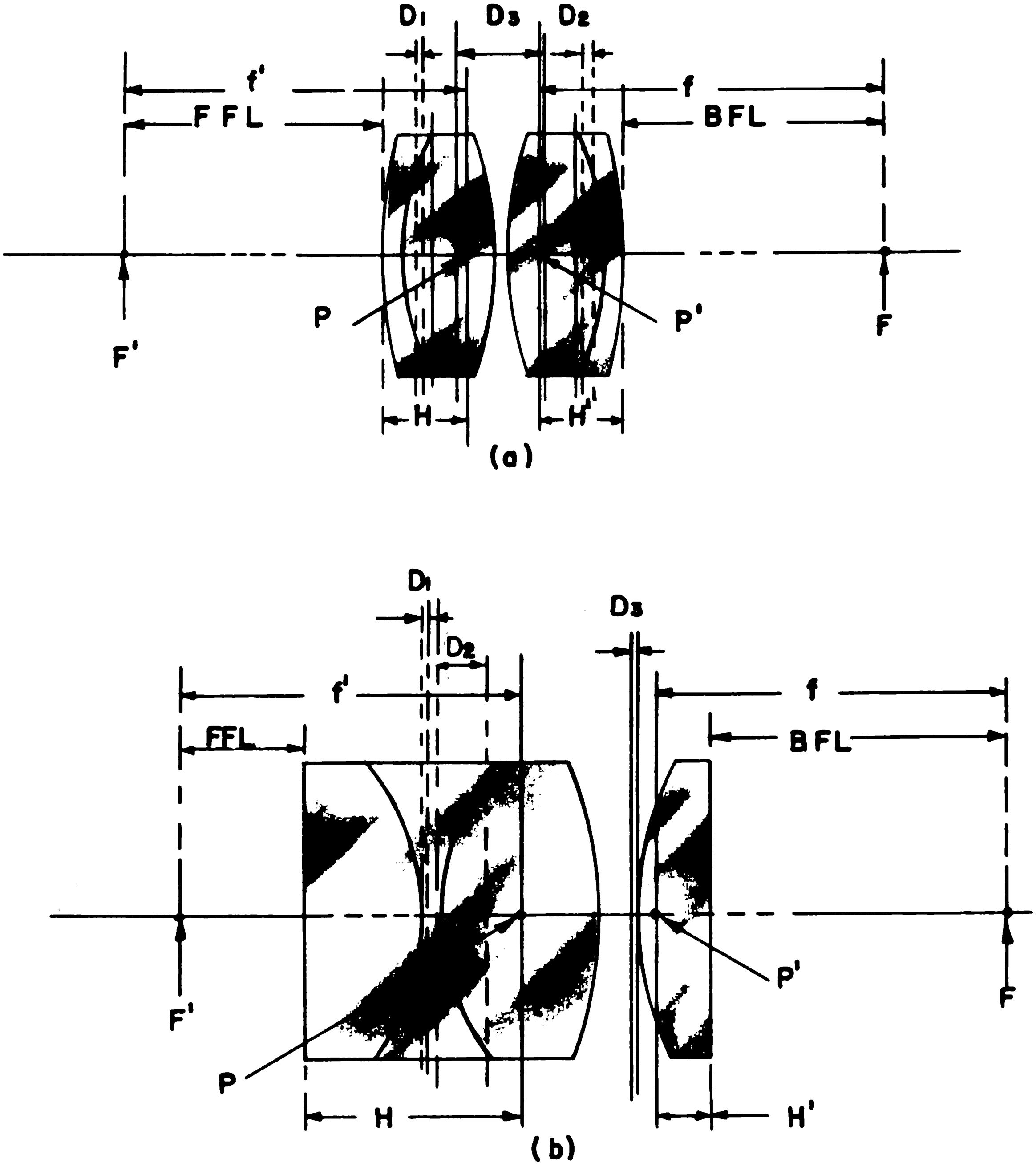
TABLE XVII

Example Figure	64 103(a)	65 103(b)	66 104(a)	67 104(b)	68 105(a)	69 105(b)
$R_1 =$	+ 2.74	+ 2.01	+ 3.9642	± ∞	— 8.4242	— 5.970
$R_2 =$	— 7.68	± ∞	+ 1.7398	— 1.3336	+ 2.0307	+ 1.895
$R_3 =$	— 4.05	— 6.60	+ 1.7398	— 1.3336	+ 2.0307	+ 1.895
$R_4 =$	+ 4.05	+ 1.88	— 2.8240	+ 1.3336	— 2.0307	— 2.164
$R_5 =$	— 4.05	— 8.11	+ 2.8240	+ 1.3336	+ 2.8101	+ 2.502
$R_6 =$	+ 4.05	+ 2.03	— 1.7398	— 2.1788	— 7.4954	— 1.759
$R_7 =$	+ 6.64	+ 2.03	— 1.7398	+ 2.0610	+ 1.9453	— 1.759
$R_8 =$	— 2.53	— 3.36	— 3.9642	± ∞	— 1.6495	— 17.057
$R_9 =$					— 1.6495	+ 1.481
$R_{10} =$					± ∞	— 1.293
$R_{11} =$						— 1.293
$R_{12} =$						± ∞
$d_1 =$	0.28	0.31	0.1250	0.6724	0.1964	0.128
$d_2 =$	0.16	0.17	0.0000	0.0000	0.0000	0.000
$d_3 =$	0.12	0.10	0.5256	0.1388	0.7858	0.469
$d_4 =$	0.36	0.43	0.0735	0.0000	0.0406	0.017
$d_5 =$	0.12	0.10	0.5256	0.9286	0.4247	0.469
$d_6 =$	0.16	0.00	0.0000	0.2228	0.0586	0.000
$d_7 =$	0.28	0.31	0.1250	0.4000	0.8258	0.085
$d_8 =$					0.0000	0.017
$d_9 =$					0.2026	0.554
$d_{10} =$						0.000
$d_{11} =$						0.085
$n_1 =$	1.6315	1.6116	1.6490	1.5170	1.7200	1.649
$n_2 =$	1.0000	1.0000	1.0000	1.0000	1.0000	1.000
$n_3 =$	1.6053	1.5831	1.5170	1.6170	1.5170	1.617
$n_4 =$	1.0000	1.0000	1.0000	1.0000	1.0000	1.000
$n_5 =$	1.6053	1.5234	1.5170	1.6170	1.5170	1.617
$n_6 =$	1.0000	1.0000	1.0000	1.0000	1.0000	1.000
$n_7 =$	1.6315	1.6073	1.6490	1.5725	1.6110	1.649
$n_8 =$					1.0000	1.000
$n_9 =$					1.6490	1.617
$n_{10} =$						1.000
$n_{11} =$						1.649



(a) The "Celor" type.
(b) The "Tessar" type.

Figure 103. Four lens systems.



(a) The "Ploessl" eye piece.
(b) The "Abbe" eye piece.

Figure 104. Four lens systems.

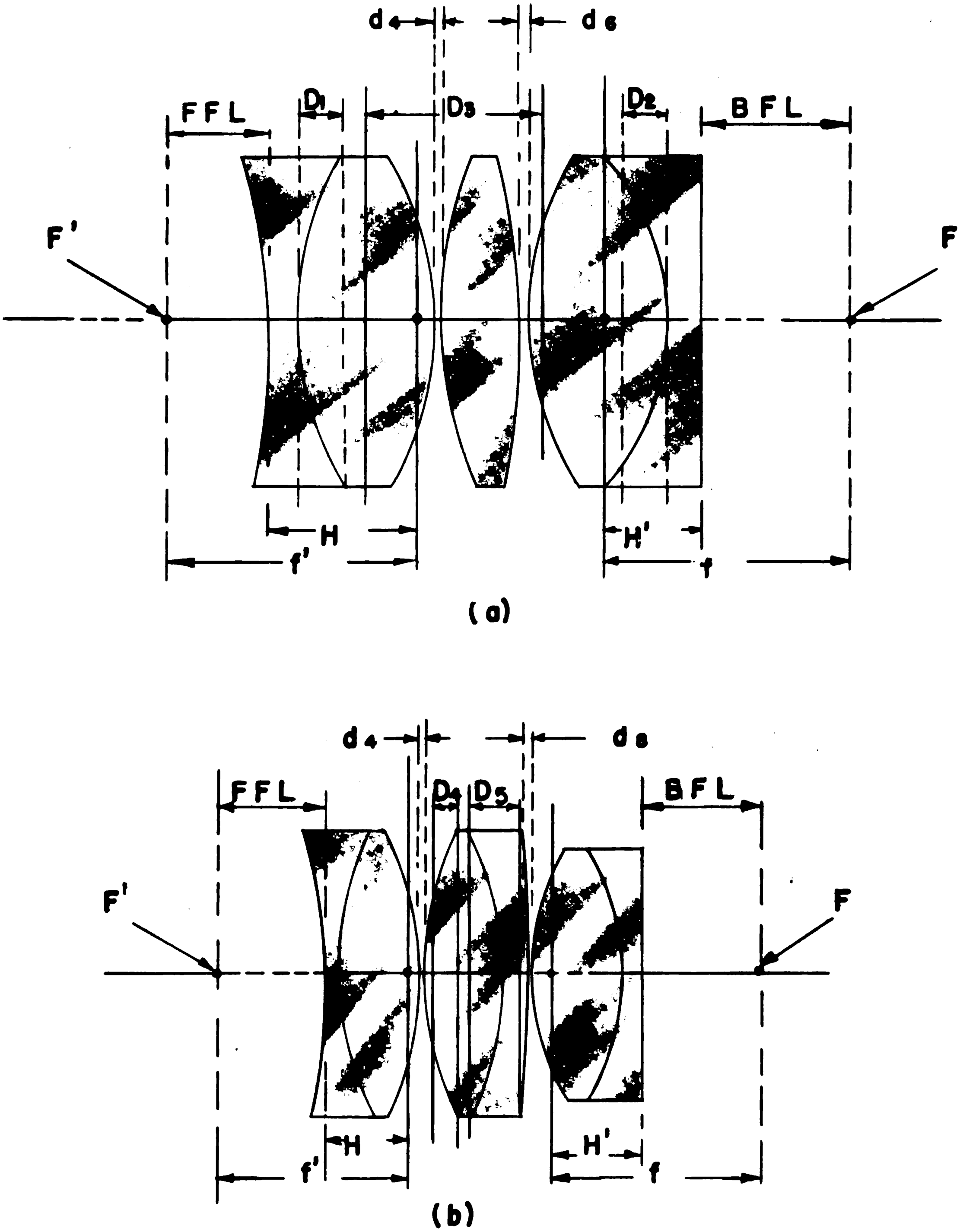


Figure 105. Wide-angle eye pieces.

Substitute these known constants into equation (390) and obtain the constants for each individual lens. In the cases of the various systems we proceed as follows:

Examples 70, 71, and 72:

Consider the first two lenses as one unit; all dimensions relating to the same will carry the subscript 12. The last two lenses form the second unit and the constants will carry the subscript 34. Finally, we combine these two units into one. Thus we write equations (390) to (397) as follows:

First Unit	Second Unit
$D_1 = d_2 - H_1' + H_3$	$D_2 = d_6 - H_5' + H_7$
$f_{12} = \dfrac{f_1 f_3}{f_1 + f_3 - D_1}$	$f_{34} = \dfrac{f_5 f_7}{f_5 + f_7 - D_2}$
$H_{12} = H_1 + \dfrac{D_1 f_1}{f_1 + f_3 - D_1}$	$H_{34} = H_5 + \dfrac{D_2 f_5}{f_5 + f_7 - D_2}$
$H_{12}' = H_3' - \dfrac{D_1 f_3}{f_1 + f_3 - D_1}$	$H_{34}' = H_7' - \dfrac{D_2 f_7}{f_5 + f_7 - D_2}$
$\text{F.F.L.} = H_{12} - f_{12}$	$\text{F.F.L.} = H_{34} - f_{34}$
$\text{B.F.L.} = H_{12}' + f_{12}$	$\text{B.F.L.} = H_{34}' + f_{34}$

For the *complete combination* we write,

$$D_3 = d_4 - H_{12}' + H_{34}$$

$$f = -f' = \frac{f_{12} f_{34}}{f_{12} + f_{34} - D_3}$$

$$H = H_{12} + \frac{D_3 f_{12}}{f_{12} + f_{34} - D_3}$$

$$H' = H_{34}' - \frac{D_3 f_{34}}{f_{12} + f_{34} - D_3}$$

$$\text{F.F.L.} = H - f \qquad\qquad \text{B.F.L.} = H' + f$$

Substituting the values obtained with equation (390) into the equations given above, we obtain the constants for each unit as well as for the complete combination.

The computed constants are given in table XVIII.

TABLE XVIII

Example Figure	73 103(a)	74 103(b)	75 104(a)
$D_1 =$	0.3250	0.4113	0.0768
$f_{12} =$	$+25.5792$	-21.8009	$+3.7885$
$H_{12} =$	-2.4531	$+3.5889$	$+0.2725$
$H_{12}' =$	-2.6096	$+2.7145$	-0.1636
F.F.L. =	-28.0323	$+25.3898$	-3.5160
B.F.L. =	$+22.9696$	-19.0864	$+3.6250$
$D_2 =$	0.3229	0.0874	0.0768
$f_{34} =$	$+13.6767$	$+6.2782$	$+3.7885$
$H_{34} =$	$+1.5416$	$+0.3098$	$+0.1636$
$H_{34}' =$	$+1.2796$	$+0.0545$	-0.2725
F.F.L. =	-12.1351	-5.9685	-3.6250
B.F.L. =	$+14.9563$	$+6.3327$	$+3.5160$
$D_3 =$	$+4.5112$	-1.9747	$+0.4006$
$f = -f' =$	$+10.0689$	$+10.1027$	$+2.0000$
$H =$	$+0.8680$	$+0.4113$	$+0.4840$
$H' =$	-0.4961	-0.8606	-0.4840
F.F.L. =	-9.2008	-9.6914	-1.5160
B.F.L. =	$+9.5727$	$+9.2421$	$+1.5160$

Example 76 (shown in fig. 104(b)):

Here we consider the cemented triplet as the first unit and the plane-convex lens as the second. The subscript 135 is attached to constants relating to the first unit. Rewriting equations (398) to (402) we substitute the known constants given in the 5th column of table XVII and also the constants computed with equation (390) and find:

$$D_1 = d_2 - H' + H_3 = 0.0421 \qquad D_2 = d_4 - H_3' + H_5 = 0.2846$$

$$J = (f_1 - D_1)f_3 + (f_1 + f_3 - D_1)(f_5 - D_2) = -0.9058$$

$$f_{135} = \frac{f_1 f_3 f_5}{J} = 4.5001$$

$$H_{135} = H_1 + \frac{\{f_3 D_2 + (f_3 + f_5 - D_2) D_1\}\ f_1}{J} = +1.2845$$

$$H_{135}' = H_5' - \frac{\{f_3 D_1 + (f_1 + f_3 - D_1) D_2\}\ f_5}{J} = +0.2219$$

$$\text{F.F.L.} = H_{135} - f_{135} = -\ 3.2157$$

$$\text{B.F.L.} = H_{135}' + f_{135} = +\ 4.7220$$

and, for the complete system:

$$D_3 = d_6 - H_{135}' + H_7 = 0.0009$$

$$f = f' = \frac{f_{135} f_7}{f_{135} + f_7 - D_3} = 2.0002$$

$$H = H_{135} + \frac{D_3 f_{135}}{f_{135} + f_7 - D_3} = +\ 1.2850 \qquad H' = H_7' - \frac{D_3 f_7}{f_{135} + f_7 - D_3} = -\ 0.2548$$

$$\text{F.F.L.} = H - f = -\ 0.7153 \qquad \text{B.F.L.} = H' + f = +\ 1.7455$$

Example 77:

In the diagram of figure 105(a) is shown the optical system for a wide-angle eye piece. The known data for this combination is given in the 6th column of table XVII. Consider the first cemented doublet as Unit I, the single center lens as Unit II, and the rear cemented doublet as Unit III. In the determination of the constants for Units I and III we write equations (393) and (397) as follows:

Unit I

$$D_1 = d_2 - H_1' + H_3 = +\ 0.2993 \qquad f_{12} = \frac{f_1 f_3}{f_1 + f_3 - D_1} = 10.4986$$

$$H_{12} = H_1 + \frac{D_1 f_1}{f_1 + f_3 - D_1} = +\ 1.5857 \qquad H_{12}' = H_3' - \frac{D_1 f_3}{f_1 + f_3 - D_1} = +\ 1.1162$$

$$\text{F.F.L.} = H_{12} - f_{12} = -\ 8.9129 \qquad \text{B.F.L.} = H_{12}' + f_{12} = +\ 11.6148$$

Unit II

The constants for this unit can be determined with equation (390) and they are given here except that now we change the subscript from 5 to 3.

$$f_3 = 4.0096 \qquad H_3 = +\ 0.0774 \qquad H_3' = -\ 0.2065$$

$$\text{F.F.L.} = -\ 3.9322 \qquad \text{B.F.L.} = +\ 3.8031$$

Unit III

$$D_2 = d_8 - H_7' + H_9 = +\ 0.2577 \qquad f_{45} = \frac{f_7 f_9}{f_7 + f_9 - D_2} = 3.3926$$

$$H_{45} = H_7 + \frac{D_2 f_7}{f_7 + f_9 - D_2} = -\ 0.0401 \qquad H_{45}' = H_9' - \frac{D_2 f_9}{f_7 + f_9 - D_2} = -\ 0.6691$$

$$\text{F.F.L.} = H_{45} - f_{45} = -\ 3.4326 \qquad \text{B.F.L.} = H_{45}' + f_{45} = +\ 2.7235$$

Eye Piece (Complete)

From equation (390):

$$D_3 = d_4 - H_{12}' + H_5 = -\,0.9982 \qquad D_4 = d_6 - H_5' + H_{45} = +\,0.2251$$

From equations (398) to (402) we obtain,

$$J = (f_{12} - D_3)f_3 + (f_{12} + f_3 - D_3)(f_{45} - D_4) = 95.2135$$

$$f = -f' = \frac{f_{12}f_3f_{45}}{J} = 1.500$$

$$H = H_{12} + \frac{[f_3D_4 + (f_3 + f_{45} - D_4)D_3]\,f_{12}}{J} = +\,0.8953$$

$$H' = H_{45} - \frac{[f_3D_3 + (f_{12} + f_3 - D_3)D_4]\,f_{45}}{J} = -\,0.6508$$

$$\text{F.F.L.} = H - f = -\,0.6046 \qquad \text{B.F.L.} = H' + f = +\,0.8491$$

Table XIX lists this data for the wide-angle eye piece shown in figure 105(a).

TABLE XIX

CONSTANTS	EYE PIECE	UNIT I	UNIT II	UNIT III
E.F.L. =	1.5000	10.4986	4.0096	3.3926
F.F.L. =	— 0.6046	— 8.9129	— 3.9322	— 3.4326
B.F.L. =	+ 0.8491	+ 11.6148	+ 3.8031	+ 2.7235
H =	+ 0.8953	+ 1.5857	+ 0.0774	— 0.0401
H′ =	— 0.6508	+ 1.1162	— 0.2065	— 0.6691

Example 78:

In the diagram of figure 105(b) is shown a wide-angle eye piece consisting of three cemented doublets. The known constants of this system are given in the last column of table XVII. Divide the system into three units as follows:

UNIT I = FRONT DOUBLET UNIT II = CENTER DOUBLET

UNIT III = REAR DOUBLET

In computing the constants for each unit and also for the complete eye piece, substitute the various computed constants first into equations (393) to (397) and then into equations (398) to (402). Since the subscripts change in each case the formulae are repeated here.

Unit I

$$D_1 = d_2 - H_1' + H_3 \qquad f_{12} = \frac{f_1 f_3}{f_1 + f_2 - D_1}$$

$$H_{12} = H_1 + \frac{D_1 f_1}{f_1 + f_3 - D_1} \qquad H_{12}' = H_3' - \frac{D_1 f_3}{f_1 + f_3 - D_1}$$

$$\text{F.F.L.} = H_{12} - f_{12} \qquad \text{B.F.L.} = H_{12}' + f_{12}$$

Unit II

$$D_2 = d_6 - H_5' + H_7 \qquad f_{34} = \frac{f_5 f_7}{f_5 + f_7 - D_2}$$

$$H_{34} = H_5 + \frac{D_2 f_5}{f_5 + f_7 - D_2} \qquad H_{34}' = H_7' - \frac{D_2 f_7}{f_5 + f_7 - D_2}$$

$$\text{F.F.L.} = H_{34} - f_{34} \qquad \text{B.F.L.} = H_{34}' + f_{34}$$

Unit III

$$D_3 = d_{10} - H_9' + H_{11} \qquad f_{56} = \frac{f_9 f_{11}}{f_9 + f_{11} - D_3}$$

$$H_{56} = H_9 + \frac{D_3 f_9}{f_9 + f_{11} - D_3} \qquad H_{56}' = H_{11}' - \frac{D_3 f_{11}}{f_9 + f_{11} - D_3}$$

$$\text{F.F.L.} = H_{56} - f_{56} \qquad \text{B.F.L.} = H_{56}' + f_{56}$$

Eye Piece (Complete)

$$D_4 = d_4 - H_{12}' + H_{34} \qquad D_5 = d_8 - H_{34}' + H_{56}$$

$$J = (f_{12} - D_4) f_{34} + (f_{12} + f_{34} - D_4)(f_{56} - D_5)$$

$$f = -f' = \frac{f_{12} f_{34} f_{56}}{J}$$

$$H = H_{12} + \frac{\{f_{34} D_5 + (f_{34} + f_{56} - D_5) D_4\}\ f_{12}}{J}$$

$$H' = H_{56}' - \frac{\{f_{34} D_4 + (f_{12} + f_{34} - D_4) D_5\}\ f_{56}}{J}$$

$$\text{F.F.L.} = H - f \qquad \text{B.F.L.} = H' + f$$

The results of this computation are listed in table XX.

TABLE XX

CONSTANTS	EYE PIECE	UNIT I	UNIT II	UNIT III
E.F.L. =	1.247	5.806	3.779	2.529
F.F.L. =	—0.660	—5.204	—3.750	—2.551
B.F.L. =	0.707	6.067	3.464	2.116
H =	+0.587	+0.602	+0.029	—0.022
H′ =	—0.540	+0.261	—0.315	—0.413

If the combination consists of more than three cemented doublets, i.e., four doublets, combine the first two into one and the other two into a second unit. In all cases substitute the given constants into equation (390) and, after the constants of each individual lens have been computed, substitute the same into equations (393) to (397). This holds good for each doublet as well as for each unit and also for the entire combination. The subscripts will change in each case, but a glance at the equations given in the preceding example should help overcome any difficulties.

PART 5. REFRACTION BY NONSPHERICAL SURFACES

81. Cartesian Lenses

In figure 106(a) is illustrated a "Cartesian" lens, that is, a lens which has at least one nonspherical surface. The diagram shows the first surface nonspherical (Cartesian) and the lens in air. In order that the Cartesian surface forms an image of a distant object free from spherical aberration at a distance L behind its vertex, the following equations must be satisfied:

$$a = \frac{nL}{n+1} \qquad b = \sqrt{a^2 - c^2} \qquad c = \frac{a}{n}$$

$$e = \frac{c}{a} = \frac{1}{n} \qquad p = \frac{(n-1)L}{n} \qquad L = \frac{(n+1)a}{n} \tag{408}$$

$$x = \frac{p \pm \sqrt{p(p - y^2/a)}}{p/a} \quad \text{(equation of the ellipse)}$$

where n is the refractive index behind the Cartesian surface; c is the numerical eccentricity of the ellipse.

If equation (408) is satisfied the rays proceeding from a distant object will come to a focus at the second focal point F of the ellipse. This point is situated at a distance L from the vertex of the lens. In order to create a Cartesian lens we grind a second surface with its center of curvature at F. It will be noted that all rays, after refraction by the Cartesian surface, will enter the spherical surface normally, hence, there will be no refraction and, therefore, the image will remain free from spherical aberration.

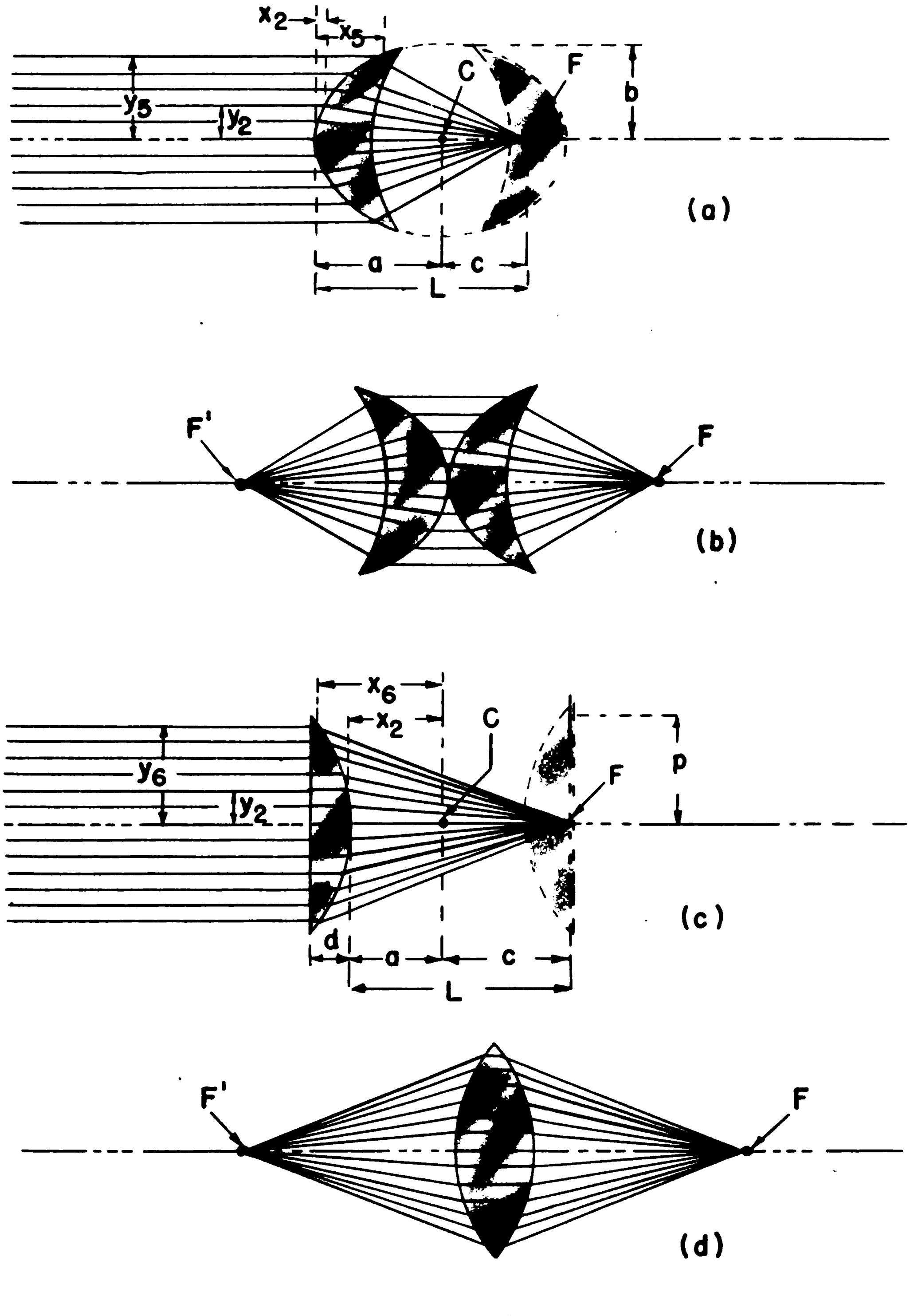


Figure 106. Cartesian lenses.

If the clear aperture of the lens is $2Y_m$, then the radius of curvature of the spherical surface is determined with the following equations:

$$\tan \alpha = \frac{Y_m}{L - x_m} \quad \text{and} \quad R_2 = \frac{Y_m}{\sin \alpha} \tag{409}$$

and the axial thickness of the lens is $d = L - R_2$ (410)

If two such lenses are placed with their Cartesian surfaces together they will form a combination which may be used as an erecting system as shown in figure 106(b).

Example 79:

A Cartesian lens is to form an image of a distant object at a distance of 4 inches behind the vertex of the Cartesian surface. The medium in front of the lens is air $(n = 1)$ and the medium between the Cartesian and the spherical surface is glass with a refractive index of 1.5165. If the aperture of the lens is equal to 3 inches, what will be the constants of the lens? We have,

$$L = 4 \text{ inches} \qquad Y_m = 1.50 \text{ inch} \qquad \text{and} \qquad n = 1.5165$$

Substituting the values into equation (408) we find,

$$a = 2.411 \qquad c = 1.590 \qquad p = 1.362$$

$$b = 1.812 \qquad e = 0.659$$

Now we divide the value of Y_m into six equal parts and substitute the various values of Y successively in the equation for x and we obtain the following results:

Y =	± 0.25	± 0.50	± 0.75	± 1.00	± 1.25	± 1.50
x =	+ 0.023	+ 0.094	+ 0.209	± 0.40	+ 0.665	+ 1.058

In this example $Y_m = 1.50$ and, substituting this value into equation (409) we find,

$$\alpha = 27°0'54'' \qquad \text{and} \qquad R_2 = +\,3.302 \text{ inches}$$

From equation (410), $d = 0.698$ inches

If the first surface of a Cartesian surface is plane as illustrated in figure 106(c) then, if the image of a distant object produced by this lens shall be free from spherical aberration, the following conditions must be satisfied:

$$a = \frac{L}{n+1}, \qquad b = a\sqrt{n^2 - 1}, \qquad c = an, \qquad d = c - a,$$

$$e = n, \qquad p = (n-1)L, \qquad L = (n+1)a, \qquad \text{and} \tag{411}$$

$$x = \frac{a}{b}\sqrt{b^2 + Y^2} \quad \text{(equation of the hyperbola)}$$

Since the rays of light proceeding from a distant object travel parallel to the optical axis they will pass through the plane surface undeviated. After a refraction by the second (Cartesian) surface of the lens the rays will emerge and come to a focus at the second focal point of the hyperbola at a distance L from the vertex of the Cartesian surface. Two such lenses placed with their plane surfaces together will form a system suitable for an erecting system. It is obvious that two such lenses can be made in one unit as illustrated in figure 106(d).

Example 80:

A Cartesian lens whose front surface is plane shall form an image of a distant object at a distance of 4 inches behind the lens. The refractive index of the medium between the two surfaces is 1.5165. What are the constants of this lens if the clear aperture is equal to 3 inches?

We have,

$L = 4$ inches, $R_1 = \pm \infty$, $n = 1.5165$ and $Y_m = 1.50$ inches

Substituting these values into equation (411) we find,

$a = 1.590$ $c = 2.411$

$b = 1.812$ $d = 0.821$ $p = 2.066$

Now we substitute the values of a and b in equation (411) for x varying successively the values of y as in the preceding example. The computed results are tabulated below:

Y =	± 0.25	± 0.50	± 0.75	± 1.00	± 1.25	± 1.50
x =	± 1.605	± 1.649	± 1.720	± 1.816	± 1.931	± 2.063

82. Cylindrical Lenses

If one or both surfaces of a lens are portions of a cylinder the lens is referred to as a *"Cylindrical Lens."*

In figure 107(a) is shown in perspective a plane-cylindrical collective lens with its axis in a vertical position. The property of this lens is that its power of converging the rays is exercised only in the horizontal meridian while no refraction will be produced in the vertical plane. Hence, the rays marked a and a′ in the diagram cannot meet the rays designated by b and b′, that is, the rays of a parallel beam of light will not come to a focus at a point f after refraction, but will focus in a straight line, the so-called *"focal line."*

The same lens, turned through an angle of 90°, is shown in figure 107(b). The axis is now horizontal, and the power of converging the rays is exercised only in the vertical plane. The rays of light proceeding from a distant object p will focus in a straight line at the focal plane P′.

Positive cylinder lenses are used a great deal for the purpose of magnifying vernier scales on instruments and scales on slide rules.

In figure 107(c) are illustrated the actions of a negative or dispersive cylinder lens.

The values of R, the radius of the cylinder, and f, the distance from the lens to the focal plane P′ are determined by applying equations (345) to (362).

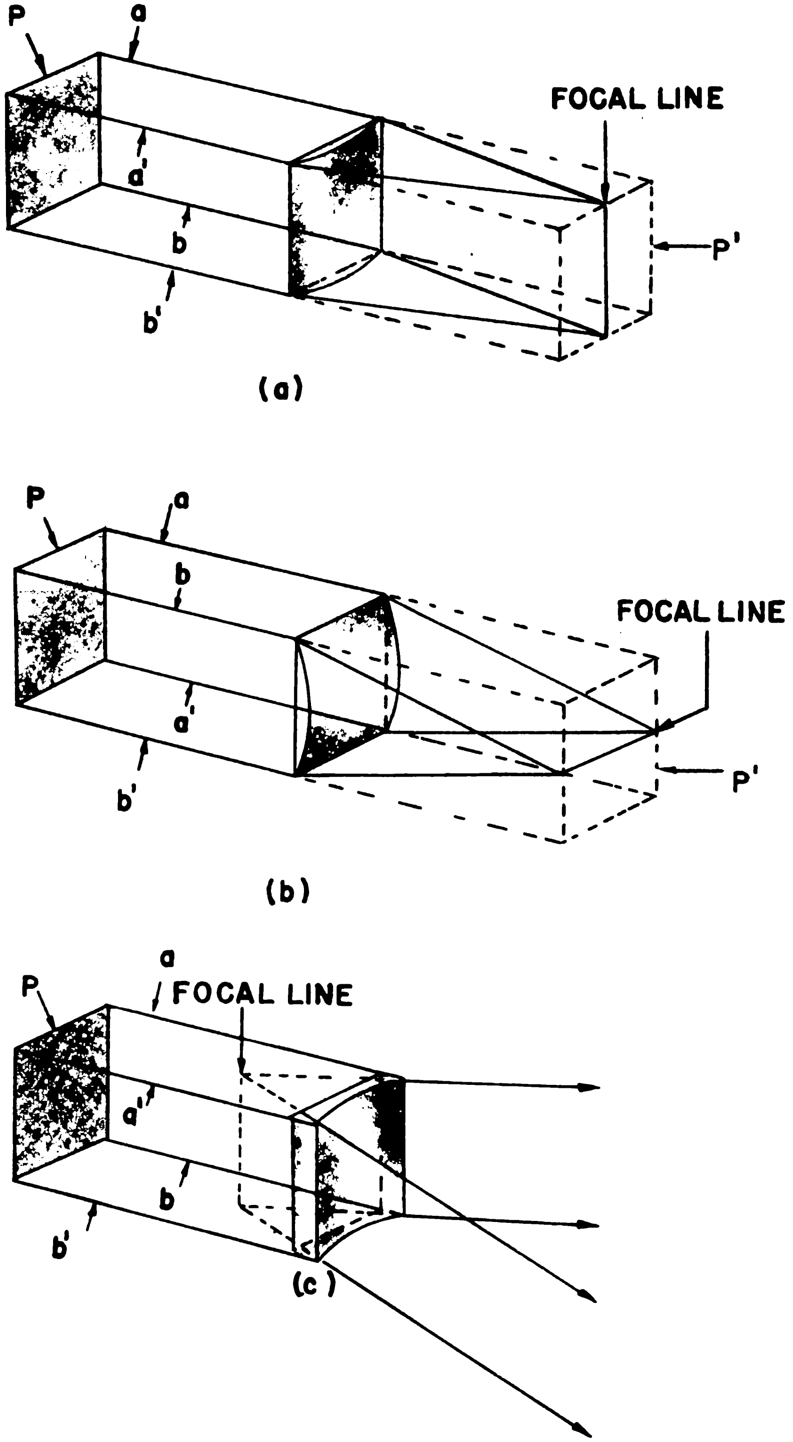


Figure 107. Cylinder lenses.

83. Annular Reflectors

Annular reflectors are employed where it is desired to observe the entire horizon simultaneously as, for example, with a periscope on a submarine.

One of the first annular reflectors used is illustrated in figure 108. The plate P is cemented or fuzed to the reflector merely for the purpose of providing means for mounting the reflector. Light from all directions incident at the silvered reflecting surface will be reflected in a downward direction. If we use an erecting system and an eye piece with this reflector we obtain a simple panoramic telescope.

A—B is a portion of a circle whose radius of curvature is R_1 with its center at C_1 as shown in figure 108(b). If this arc is rotated about the axis Y—Y′, lying in the plane of the paper, then the arc describes a curved conical surface which is termed a *"toric surface."* The traces of this surface in the plane of the paper are the arcs AB and A′B′.

If this surface is reflecting it will form an image of distant objects on the horizon which can be seen upon placing a screen at a short distance below the reflector. Let D—E be a horizontal incident ray, and C_1—E the normal at E, then the ray D—E will be reflected in a vertical direction so that $i = i'$. One of the most annoying defects of this reflector is astigmatism. This defect is due to the fact that the rays of light proceeding from a distant object will form two image points, a primary image point at F_t and a secondary point at F_s.

From the law of astigmatic reflection we have,

$$E - F_t = t' = \frac{R_1 \cos i}{2} \tag{412}$$

and

$$E - F_s = S' = \frac{R_2}{2 \cos i} \tag{413}$$

where R_2 is the second radius of the toric surface produced by rotation in the sagittal plane (see fig. 108(b)). The distance between the focal points F_t and F_s is the astigmatism caused by the reflecting surface.

If the reflector is to produce an anastigmatic image the following condition must be satisfied,

$$t' = S' = \frac{R_1 \cos i}{2} = \frac{R_2}{2 \cos i}$$

hence,

$$R_2 = R_1 \cos_2 i \tag{414}$$

and

$$\frac{R_1}{R_2} = \frac{1}{\cos^2 i} \tag{415}$$

If the astigmatism is eliminated there still remains a certain amount of spherical aberration, sine condition error, and distortion, and we have no means of eliminating

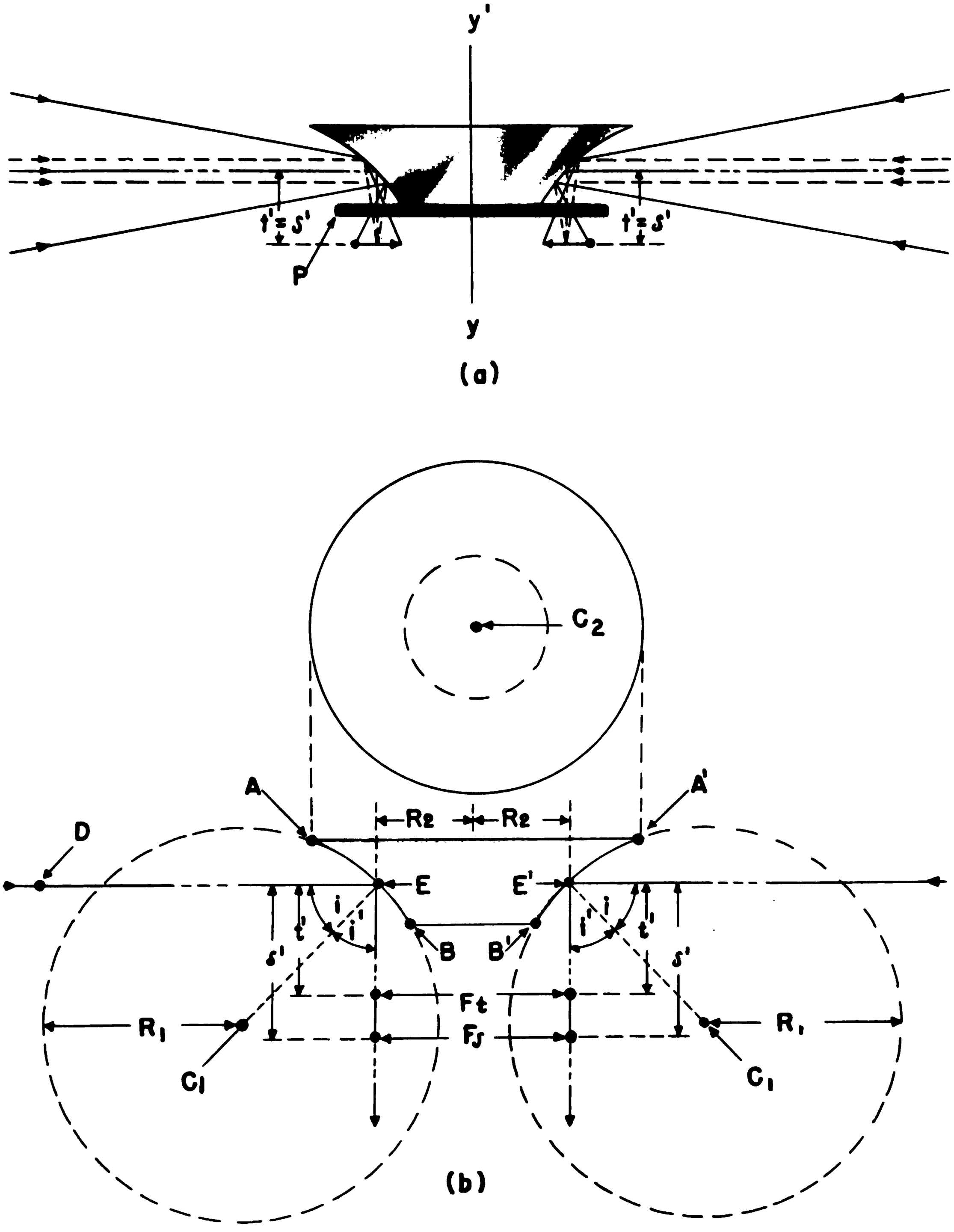


Figure 108. The annular reflector.

these errors. In the case of a periscope the angle of incidence i will always be equal to 45°, hence, $\cos^2 i = 0.5$. Substituting this value into equation (414) we have

$$R_1 = 2R_2 \tag{416}$$

Example 81:

The radius of the first reflecting surface of an annular reflector used in a periscope is equal to 2 inches, and the angle of incidence is $i = 45°$. What is the magnitude of R_2 if the image is to be free from astigmatism?

We have, $R_1 = 2$ inches and $i^2 = 0.50$

With the aid of equation (236) we find, $R_2 = R_1/2 = 1$ inch.

Substitute these values into equations (232) and (233) and find,

$$t' = S' = 0.707 \text{ inches}$$

Figure 109 illustrates the annular reflector designed by J. Humbrecht. Rays of light proceeding from a distant object on the horizon are incident at the annular reflector. After a refraction at the first surface they proceed toward the focal point F_1 and, if coming from the right, they will converge toward the focal point F_2. These points are shown in the lower diagram of figure 109. The distances between these points and the vertices are designated by s_1'. However, the rays are not permitted to reach these points but the silvered elliptical surface will reflect them in a downward direction in such a way that the rays, when projected backwards, will focus at the points F_1' and F_2'. These points are located in the image plane of the reflector and are separated from the horizontal axis by the distance s_2'. The rays will diverge from the reflector; hence, the reflector will act like a dispersive lens.

The plate P is cemented or fuzed to the reflector merely to provide means for mounting the reflector properly.

The construction of the Humbrecht annular reflector is more readily understood by a study of the lower diagram in figure 109. The points F_1 and F_1' are the focal points of the first ellipse, and the points F_2 and F_2' those of the second ellipse.

Assume that the constants R_1, n, and s_2' are given; then, in order to determine the magnitudes of the unknown constants we apply the following equations,

$$\begin{aligned} s_1' &= \frac{nR_1}{n-1} & d &= \frac{(3n-4)R_1}{3(n-1)} \\ a &= \frac{s_1' - d + s_2'}{2} & b &= a\sqrt{1-e^2} \\ e &= \frac{s_1' - d}{2a \cos \alpha} & \tan \alpha &= \frac{s_2'}{s_1' - d} \end{aligned} \tag{417}$$

In the above equations a is the length of the semi-major axis of the ellipse and b that of the semi-minor axis. e is the eccentricity of the ellipse.

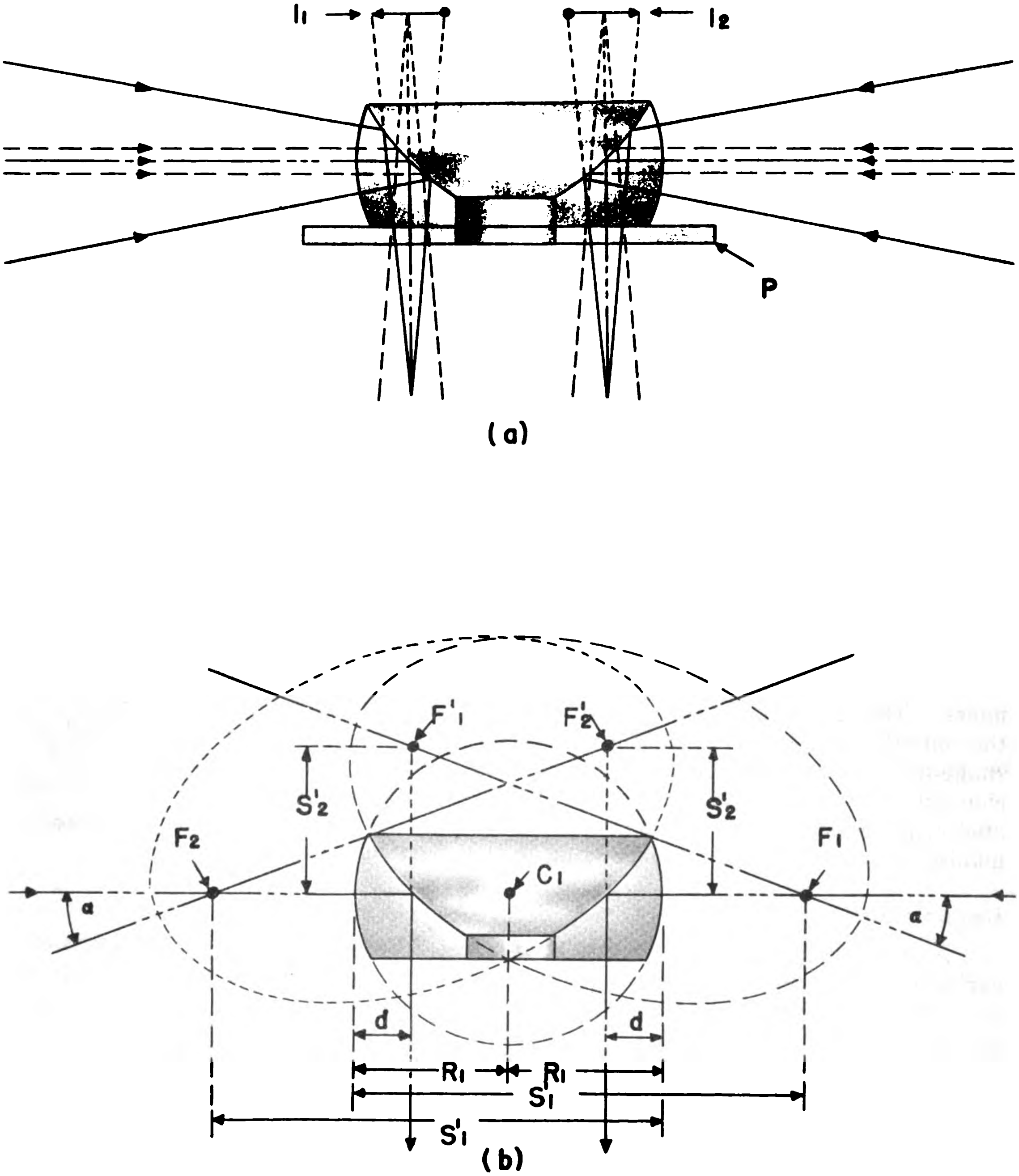


Figure 109. The annular reflector by J. Humbrecht.

Example 82:

The radius of curvature of the spherical surface of a Humbrecht annular reflector is equal to 0.75 inches, and the virtual image plane is situated at a distance of 0.735 inches above the horizontal axis of the reflector. The glass has a refractive index of 1.5170. What are the magnitudes of the unknown constants?

We have, $R_1 = 0.75$, $s_2' = 0.735$, and $n = 1.5170$

Substitute these values into equation (417) and find:

$s_1' = 2.201$ $\tan \alpha = 0.379844$ $e = 0.775$ $a = 1.335$

$d = 0.266$ $\alpha = 20°47'56''$ $b = 0.844$

PART 6. SPECIAL COMBINATIONS

84. Variable Wedge by Boslovich

In figure 110 are shown diagrams of a lens compensator designed by Boslovich in 1777. This system consists of two lenses of equal but opposite focal lengths. Both lenses are made of the same glass and their spherical surfaces are nearly in contact. The diagram to the left shows the axis AI of the instrument in coincidence with the axes of both lenses and, hence, there will be no deviation. If we turn the positive lens about the center C through an angle θ, the combination has been converted into a wedge and, consequently, the ray will be refracted and, upon emerging from the system, it will include the angle ε with the axis AI. If the compensator (in the position shown in the left diagram) is placed in front of a coincidence range finder objective there will be no deviation and we say, the compensator is in infinity position. A ray of light proceeding from a target at a finite distance r, will include the angle ε with the axis of the range finder. This angle is called a *"parallactic angle."* By means of a mechanical device the collective lens is now rotated about the center C until the two images formed in the range finder are in coincidence, that is, the ray emerging from the compensator will coincide with the axis. Turning the collective lens will also move a pointer along a graduated range scale. When the images are in coincidence the pointer will be opposite the number giving the range corresponding to that particular angle ε.

Example 83:

A range finder has a base length, B, of 5 feet. Ranges, r, between the limits of 500 yards and 5000 yards are to be measured. The refractive index n of the glass is equal to 1.517. What are the magnitudes of the angles ε and θ:

We have,

$B = 5$ feet, $r = 500$ yards $= 1500$ ft, and $r = 5000$ yards $= 15000$ ft.

Substitute these values into the following equations,

$$\tan \varepsilon = \frac{B}{r} \tag{418}$$

and

$$\theta = \frac{\varepsilon}{n - 1} \tag{419}$$

$r = 1500$ ft.	15000 ft.
$\varepsilon = 0.1910^\circ = 11.46'$	$0.0191^\circ = 1.146'$
$\theta = \frac{11.46}{.517} = 22.17'$	$\frac{1.146}{.517} = 2.217'$

85. Sliding Lens Compensator

The sliding lens compensator has been described for years in various technical papers but in each case the name of the designer has been omitted. This compensator also consists of two lenses of equal but opposite focal lengths, and both lenses are made of the same glass. In some cases the plane surfaces are nearly in contact, in others the lenses are separated by a short distance with their spherical surfaces facing each other. This type of compensator is illustrated in figure 111.

The diagram to the left shows the compensator in its infinity position since the axis AI of the instrument coincides with the axes of both lenses and there will be no deviation. If we slide the positive lens so that the distance between its axis AC is separated from the axis AI by the distance h, then

$$h = f \tan \varepsilon \tag{420}$$

where f is the focal length of the collective lens.

Example 84:

This compensator is to be used in the range finder considered in the preceding example. The focal length of the positive lens is 150 inches. To what extent must the lens be displaced if the ranges of 1500 feet and 15000 feet are to be measured?

The value of ε has already been computed in the last example and with the aid of equation (420) we find the values for h.

The results are,

$r = 1500$ ft.	15000 ft.
$\varepsilon = 11.46'$	$1.146'$
$h = 0.50$ inch	0.05 inch

86. Bi-Prismatic Lens

The bi-prismatic lens may be thought of as being made up of a Fresnal bi-prism cemented to a plane-convex lens. However, in practice, the system is made as a unit as illustrated in figure 112. This system is used in the 9-foot Base Range Finder made by Barr & Stroud of England. A short distance from the bi-prismatic lens there is located the ocular prism system (not shown in the diagram) in which are formed two halves of the images. However, the centers of these images are separated by an angular distance of 14 degrees with respect to the lens. It is the duty of this bi-prismatic lens to bring the centers of these images into coincidence and then transmit converging rays of light towards the eye lens.

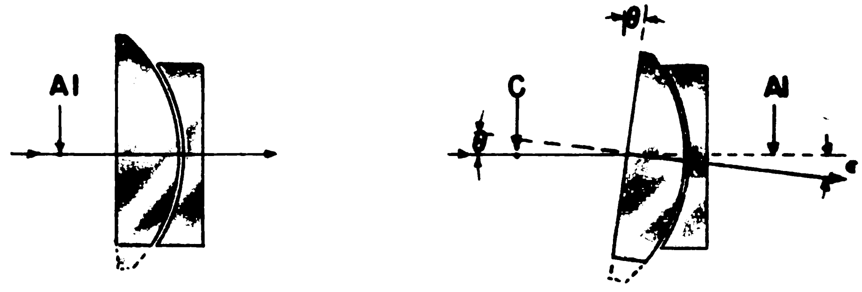


Figure 110. The variable wedge.

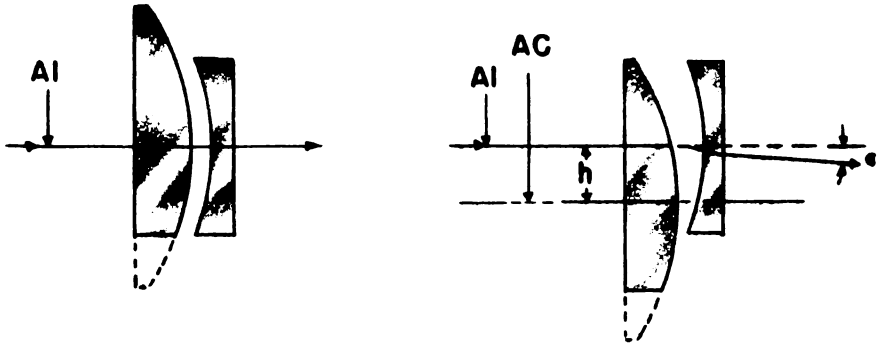


Figure 111. The sliding lens compensator.

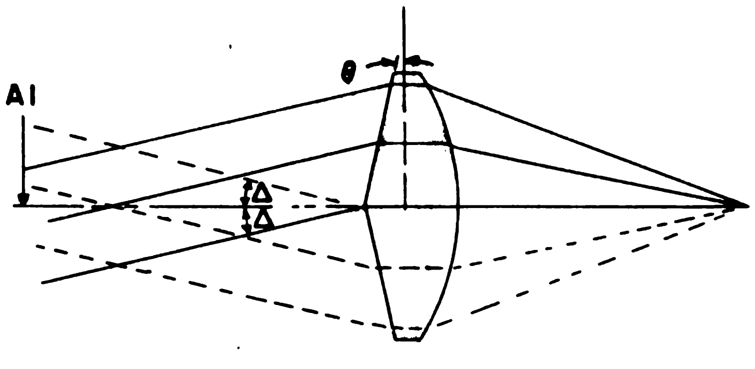


Figure 112. The bi-prismatic lens.

Example 85:

The rays proceeding from the centers of the images formed inside the ocular prism include angles of 7° with the axis AI. The index of refraction of the glass is 1.517. What is the magnitude of the refracting angle θ if the centers of the images are to be united?

We have, $\Delta = 7°$ and $n = 1.517$

Substitute these values in the following equation and find

$$\sin \theta = \frac{\sin \Delta}{\sqrt{(n^2 + 1) - 2n \cos \Delta}} = 0.2263431 \tag{421}$$

hence, $\theta = 13.0819° = 13°4'54.8''$

SECTION VI

TRIGONOMETRIC RAY TRACING FORMULAE

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87. Introduction

The application of the trigonometric ray tracing formulae is absolutely necessary in computing the magnitudes of the residual aberrations of an optical component, or a complete telescope system.

About 1866, L. Seidel published the first set of trigonometric ray tracing formulae for computing the errors caused by any number of spherical surfaces. In their "Handbuch der Angewandten Optik," published in 1891, A. Steinheil and E. Voit applied these formulae to a great number of numerical examples and for any object distance. They also included the correction of the errors in telescope objectives by means of these formulae. In 1918, J. W. French translated this work into English under the title, "Applied Optics—The Computation of Optical Systems." It is published in two volumes.

In the publications mentioned above all computations were made with the use of logarithms and it became necessary to rearrange the formulae in order to make them more convenient for the use with the calculating machines. The new formulae will be given in the following pages of this section. The formulae for the sagittal and the tangential rays have been successfully rearranged thus reducing the computations to a minimum.

88. Abbreviations and Sign Conventions

d_v = the axial thickness of the lens, measured on the axis.

d_v' = the oblique thickness, or the length of the light-path in the vth lens.

h_v = the distance from the axis to the point of incidence at the vth surface. It is positive when measured above the axis and vice versa.

s_v = distance from the vth surface to the respective object plane.

s_v' = distance from the vth surface to the respective image plane.
σ_v and σ_v', respectively, are the reciprocal values of these constants. These values are considered negative when object (image) is situated to the left of the vth surface and vice versa.

i_v = the angle of incidence of the ray at the vth surface.

i_v' = the angle of refraction at the vth surface.
i_v and i_v' are considered positive when the ray is turned clockwise in order to coincide with the normal and vice versa.

x_1 = distance from the vth surface of the lens to the diaphragm.

x_v' = distance from the vth surface to the corresponding diaphragm image.
Both constants are considered negative when the diaphragm (image) are located to the left of the vth surface and vice versa.

y = the height of the object.

y' = the ideal height of the image.

y_k' = the actual height of the image.

y_v = the heights of incidence of the principal ray and the upper and lower rim rays at the vth surface.

These values are positive when measured above the axis and vice versa.

α_v = the angle included by the incident ray and the axis.

α_v' = the angle included by the refracted ray and the axis.

Both angles are considered positive if the ray is turned counterclockwise in order to coincide with the axis and vice versa.

SUBSCRIPTS at the lower left side of a constant have reference to a particular ray. For example,

Subscript L refers to the Lower Rim Ray.

Subscript p refers to the Paraxial Ray.

Subscript Pr refers to the Principal Ray.

Subscript r refers to the Rim or Marginal Ray.

Subscript U refers to the Upper Rim Ray.

Subscript S refers to the Sagittal Ray.

Subscript t refers to the Tangential Ray.

Subscript z refers to the Zero Ray.

The subscript k at the lower right side of a constant corresponds to the serial number of the last surface of the optical system.

89. Formulae

On the following pages are given the trigonometric formulae for the various rays. They are arranged as follows:

The *"Zero Ray"* is a ray coming from a distant object traveling parallel to and very close to the optical axis.

The *"Paraxial Ray"* is a ray coming from an object situated at a finite distance from the lens. The height of incidence at the lens is infinitely small. However, for clarity the ray is shown in the diagrams at a finite distance from the axis.

The *"Rim Ray,"* or the *"Marginal Ray"* is a ray coming from an object that will enter the lens at a considerable distance from the axis.

The *"Principal Ray"* comes from an object point at a distance y from the axis. It will include the angle α_0 with the axis.

The *"Upper Rim Ray"* and the *"Lower Rim Ray"* also come from an object point off the axis but they are incident at the lens at a considerable distance from the point of incidence for the principal ray.

The *"Sagittal Ray"* travels parallel to and very close to the principal ray. Both rays travel in the sagittal plane.

The *Tangential Ray"* also travels parallel to and very close to the principal ray. However, in this case both rays travel in the tangential plane.

The *"Skew Ray"* comes from an object point lying either below or above the axis and also to the right or left of the axis.

With the exception of the Skew Ray, the formulae are given for plane as well as spherical surfaces.

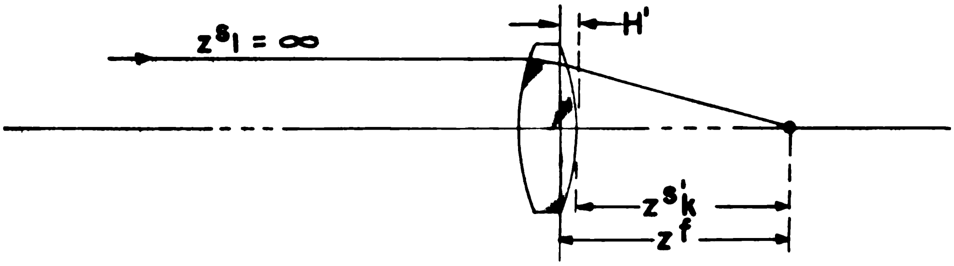


Figure 113.

The Zero Ray

$$a_1 = 0 \tag{422}$$

$$a_\nu = \frac{n_{\nu-1}\,\sigma_\nu}{n_\nu} \tag{423}$$

$$b_\nu = \frac{n_\nu - n_{\nu-1}}{n_\nu R_\nu} \tag{424}$$

$$\sigma_\nu' = a_\nu + b_\nu \tag{425}$$

$$\sigma_{\nu+1} = \frac{\sigma_\nu'}{1 - d_\nu \sigma_\nu'} \tag{426}$$

$$h_\nu = (1 - d_{\nu-1}\,\sigma_{\nu-1})\,h_{\nu-1} \tag{427}$$

$$\alpha_k = h_k \sigma_k' \tag{428}$$

$${}_z f = \text{E.F.L.} = \frac{h_1}{\alpha_k} \tag{429}$$

$${}_z s_k' = \frac{1}{\sigma_k'} \tag{430}$$

$$H' = {}_z s_k' - {}_z f \tag{431}$$

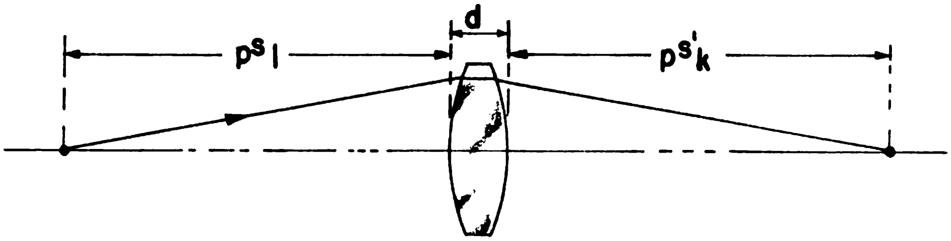


Figure 114.

The Paraxial Ray

The object distance $_ps_1$ is finite, hence σ_1 is also known. The formulae are identical with the ones for the zero ray, however, for convenience, they are repeated below.

$$a_v = \frac{n_{v-1}\,\sigma_v}{n_v} \qquad b_v = \frac{n_v - n_{v-1}}{n_v R_v}$$

$$\sigma_v' = a_v + b_v \qquad \sigma_{v+1} = \frac{\sigma_v'}{1 - d_v \sigma_v'}$$

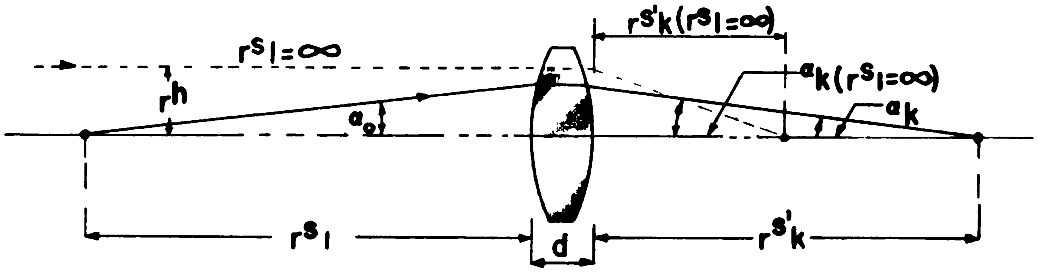


Figure 115.

The Rim Ray

Object at Finite Distance

$$\sin i_v = \frac{({}_r s_v - R_v) \sin \alpha_{v-1}}{R_v} \tag{432}$$

$$\sin i_v' = \frac{n_{v-1} \sin i_v}{n_v} \tag{433}$$

$$\alpha_v = \alpha_{v-1} + i_v - i_v' \tag{434}$$

$${}_r s_v' = \frac{R_v \sin i_v'}{\sin \alpha_v} + R_v \tag{435}$$

$${}_r s_v = {}_r s'_{v-1} - d_{v-1} \tag{436}$$

$$\text{B.F.L.} = {}_r s_k' \tag{437}$$

$$\omega_v = \alpha_{v-1} + i_v = \alpha_v + i_v' \tag{438}$$

$${}_r h_v = R_v \sin \omega_v \tag{439}$$

Object at Infinity

$$\sin i_1 = \frac{{}_r h_1}{R_1} \tag{440}$$

$${}_r f = \frac{{}_r h_1}{\sin \alpha_k} \tag{441}$$

NOTE: The rim ray for object at infinite distance is shown by dotted line.

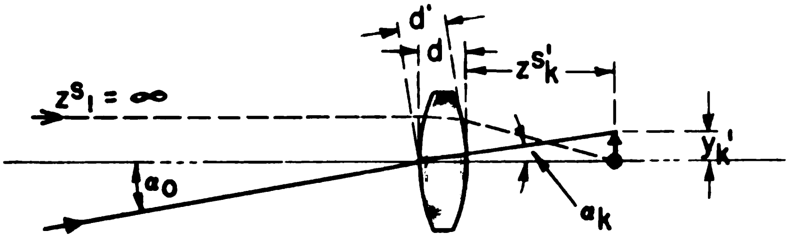


Figure 116.

The Principal Ray ($x_1 = 0$)

Object at Infinity

$$\sin i_1 = - \sin \alpha_0 \tag{442}$$

$$\sin i_v = \frac{(x_v - R_v) \sin \alpha_{v-1}}{R_v} \tag{443}$$

$$\sin i_v' = \frac{n_{v-1} \sin i_v}{n_v} \tag{433}$$

$$\alpha_v = \alpha_{v-1} + i_v - i_v' \tag{434}$$

$$x_v' = \frac{R_v \sin i_v'}{\sin \alpha_v} + R_v \tag{444}$$

$$x_{v+1} = x_v' - d_v \tag{445}$$

$$\omega_v = \alpha_{v-1} + i_v = \alpha_v + i_v' \tag{438}$$

$$y_v = R_v \sin \omega_v \tag{446}$$

$$y_k' = (x_k' - {}_s s_k') \tan \alpha_k \tag{447}$$

$$d_v' = \frac{y_v - y_{v+1}}{\sin \alpha_v} \tag{448}$$

$$y' = - {}_s f \tan \alpha_0 \tag{449}$$

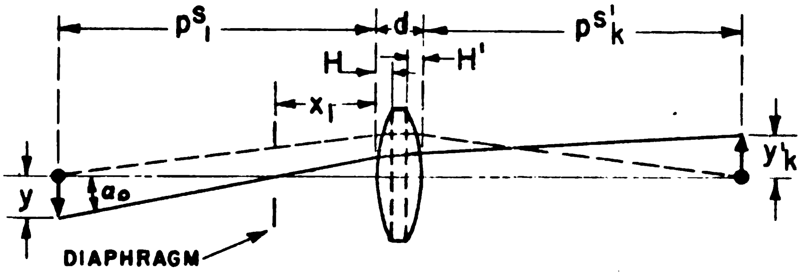


Figure 117.

The Principal Ray (x_l is given)

Object at Finite Distance

Apply successively the following equations given above: (443), (433), (434), (444), (445), (438), (446) and (448). Then compute the values of y_k' and y' as follows:

$$y_k' = (x_k' - {}_ps_k') \tan \alpha \tag{450}$$

$$y' = \frac{({}_ps_k' - H')\, y}{{}_ps_l - H} \tag{451}$$

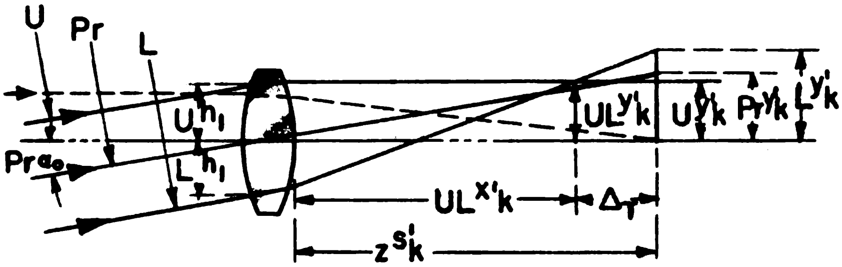


Figure 118.

Upper (U) and Lower (L) Rim Ray

The upper and lower rim rays are usually traced in conjunction with the principal ray.

Object at Infinity

The initial constants are computed as follows:

$$\sin {}_{U}\varepsilon_1 = \frac{{}_{U}h_1}{R_1} \qquad \sin {}_{L}\varepsilon_1 = \frac{{}_{L}h_1}{R_1} \qquad \sin {}_{Pr}\varepsilon_1 = 0$$

where ε_ν is the angle the normal to the surface at the point of incidence makes with the optical axis

$$ {}_{U}i_1 = {}_{U}\varepsilon_1 - \alpha_0 \qquad {}_{Pr}i_1 = {}_{Pr}\varepsilon_1 - \alpha_0 \qquad {}_{L}i_1 = {}_{L}\varepsilon_1 - \alpha_0$$

Now apply successively the equations given on the preceding page as follows: (443), (433), (434), (444), (445), (438), (446), (447), (448) and (450). Then apply the following formulae:

$${}_{UL}x_k' = \frac{{}_{U}x_k' \tan {}_{U}\alpha_k - {}_{L}x_k' \tan {}_{L}\alpha_k}{\tan {}_{U}\alpha_k - \tan {}_{L}\alpha_k} \tag{452}$$

$$\Delta T = {}_{UL}x_k' - {}_{z}s_k' \tag{453}$$

$${}_{UL}y_k' = ({}_{U}x_k' - {}_{UL}x_k') \tan {}_{U}\alpha_k = ({}_{L}x_k' - {}_{UL}x_k') \tan {}_{L}\alpha_k \tag{454}$$

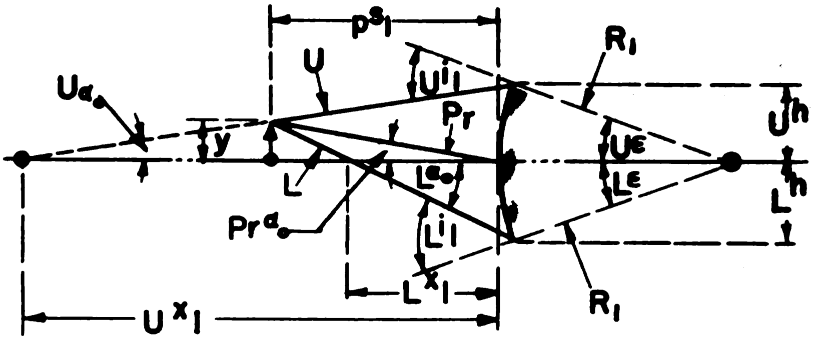


Figure 119.

Object at Finite Distance

Computing the initial constants.

$$\tan {_U\alpha_o} = \frac{_Uh - y}{_ps_1} \qquad \tan {_{Pr}\alpha_o} = -\frac{y}{_ps_1}$$

$$\tan {_L\alpha_o} = \frac{_Lh - y}{_ps_1} \qquad y = {_ps_1} \tan {_{Pr}\alpha_o}$$

$$_Ux_1 = \frac{_Uh}{\tan {_U\alpha_o}} \qquad _{Pr}x_1 = 0 \qquad _Lx_1 = \frac{_Lh}{\tan {_L\alpha_o}}$$

$$\sin {_U\epsilon_1} = \frac{_Uh}{R_1} \qquad \sin {_{Pr}\epsilon} = 0 \qquad \sin {_L\epsilon_1} = \frac{_Lh}{R_1}$$

$$_Ui_1 = {_U\epsilon} - {_U\alpha_o} \qquad _{Pr}i_1 = {_{Pr}\alpha_o} \qquad _Li_1 = {_L\epsilon_1} - {_L\alpha_o}$$

Now apply the equations listed for the rim rays with $_zs_1 = \pm \infty$. Equation (453) will read

$$\Delta T = {_{UL}x_k}' - {_ps_k}' \tag{455}$$

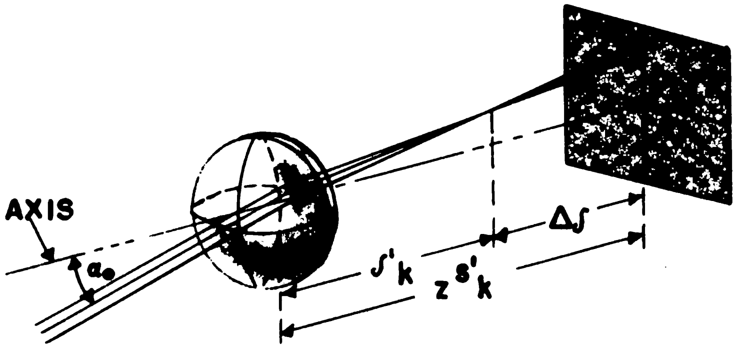


Figure 120.

The Sagittal Ray (${}_s s_1 = S_1 = \infty$)

The values of i_v, i_v', α_v, ω_v, and d_v' are taken from the principal ray trace.

$${}_s a_1 = 0 \tag{456}$$

$${}_s a_v = \frac{n_{v-1}\,{}_s\sigma_v}{n_v} \tag{457}$$

$${}_s b_v = \frac{n_v \cos i_v' - n_{v-1} \cos i_v}{n_v R_v} \tag{458}$$

$${}_s\sigma_v' = {}_s a_v + {}_s b_v \tag{459}$$

$${}_s\sigma_{v+1} = \frac{{}_s\sigma_v'}{1 - d_v'\,{}_s\sigma_v'} \tag{460}$$

$$S_v' = \frac{1}{{}_s\sigma_v'} \tag{461}$$

$$T_v = (1 - \cos \omega_v) R_v \tag{462}$$

$$\Delta S = S_k' \cos \alpha_k + (T_k - {}_s s_k') \tag{463}$$

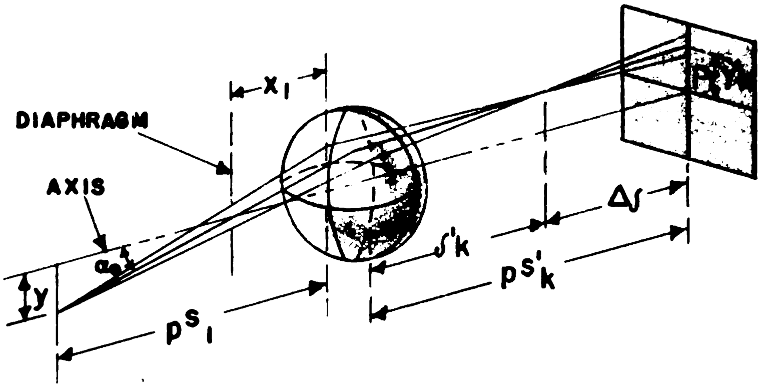


Figure 121.

Object at Finite Distance

$$S_1 = \frac{{}_p s_1 - T_1}{\cos \alpha_0} \quad \text{(when diaphragm is used)} \qquad (464)$$

$$S_1 = \frac{{}_p s_1}{\cos \alpha_0} \quad (\text{when } x_1 = 0) \qquad (465)$$

Now apply successively equations (457), (458), (459), (460), (461), and (462). Equation (463) is now written as follows:

$$\Delta S = S_k' \cos \alpha_k + (T_k - {}_p s_k') \qquad (466)$$

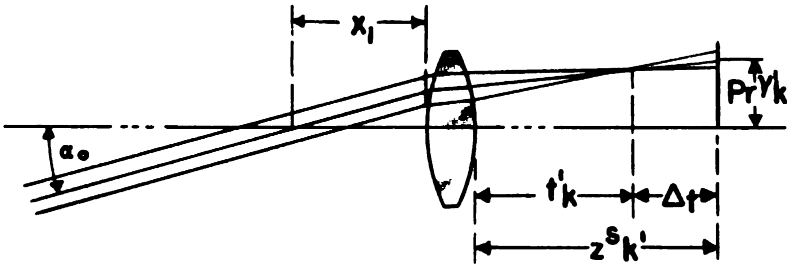


Figure 122.

The Tangential Ray ($_s s_1 = t_1 = \infty$)

The values of i_v, i_v', α_v, ω_v, and d' are taken from the principal ray trace.

$$_t a_1 = 0 \tag{467}$$

$$_t a_v = \frac{n_{v-1} \cos^2 i_v \; _t\sigma_v}{n_v \cos^2 i_v'} \tag{468}$$

$$_t b_v = \frac{_s b_v}{\cos^2 i_v'} \tag{469}$$

$$_t\sigma'_v = {_t a_v} + {_t b_v} \tag{470}$$

$$_t\sigma_{v+1} = \frac{_t\sigma_v'}{1 - d' \; _t\sigma_v'} \tag{471}$$

$$t_v' = \frac{1}{_t\sigma_v'} \tag{472}$$

$$\Delta t = t_k' \cos \alpha_k + (T_k - {_s s_k'}) \tag{473}$$

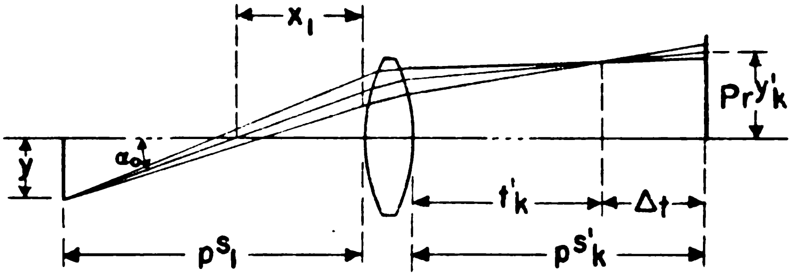


Figure 123.

Object at Finite Distance

$$t_1 = \frac{{}_p s_1 - T_1}{\cos \alpha_0} \quad \text{(when diaphragm is used)} \tag{474}$$

$$t_1 = \frac{{}_p s_1}{\cos \alpha_0} \quad \text{(when } x_1 = 0\text{)} \tag{475}$$

Apply now successively equations (468), (469), (470), (471), and (472). Equation (473) will be written as follows:

$$\Delta t = t_k' \cos \alpha_k + (T_k - {}_p s_k') \tag{476}$$

The Skew Ray

Lange's Formulae

In these formulae the signs of cos i_v and cos i_v' are the opposite from those of the corresponding R_v in the principal ray trace. The values of R_v and U_v are considered positive except in equation (493) where the original sign will be used.

$$\tau_v = n_v \cos i_v' - n_{v-1} \cos i_v \tag{477}$$

$$\epsilon_{v+1} = \epsilon_v + x_v \tau_v \rho_v \tag{478}$$

$$\zeta_{v+1} = \zeta_v + y_v \tau_v \rho_v \tag{479}$$

$$\eta_{v+1} = \eta_v + z_v \tau_v \rho_v \tag{480}$$

$$\epsilon_v^2 + \zeta_v^2 + \eta_v^2 = n^2_{v-1} \quad \text{(CHECK FORMULA)} \tag{481}$$

$$U_v = \sqrt{(R_v + K_{v+1})^2 - 2K_{v+1}(R_v - x_v)} \quad \text{where,} \quad K_{v+1} = R_v - R_{v+1} - d_v \tag{482}$$

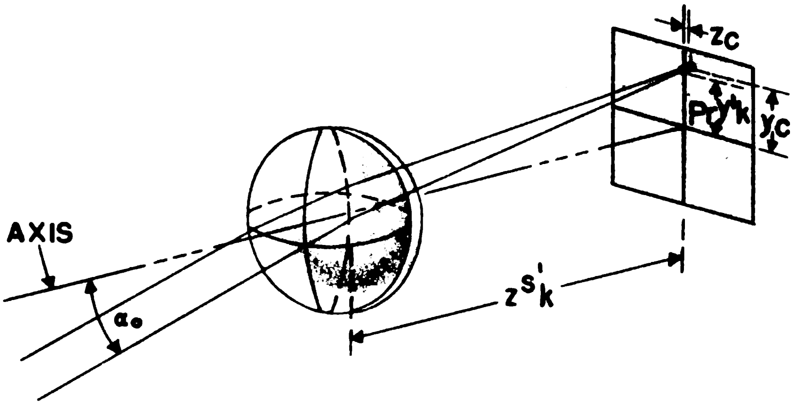


Figure 124.

$$\cos \beta_v = \frac{\varepsilon_{v+1} K_{v+1} + n_v R_v \cos i_v'}{n_v U_v} \tag{483}$$

$$\sin i_{v+1} = \frac{U_v \sin \beta_v}{R_{v+1}} \tag{484}$$

$$\sin i'_{v+1} = \frac{n_v \sin i_{v+1}}{n_{v+1}} \tag{485}$$

$$V_{v+1} = \frac{R_{v+1} \cos i_{v+1} - U_v \cos \beta_v}{n_v} \tag{486}$$

$$x_{v+1} = x_v + K_{v+1} + V_{v+1}\, \varepsilon_{v+1} \tag{487}$$

$$y_{v+1} = y_v + V_{v+1}\, \zeta_{v+1} \tag{488}$$

$$z_{v+1} = z_v + V_{v+1}\, \eta_{v+1} \tag{489}$$

$$c = {}_z s_k' - R_k \tag{490}$$

$$\left.\begin{aligned} x_v^2 + y_v^2 + z_v^2 &= R_v^2 \qquad &(491)\\ \frac{x_v\, \varepsilon_v}{R_v} + \frac{y_v\, \zeta_v}{R_v} + \frac{z_v\, \eta_v}{R_v} &= n_{v-1} \cos i_v \qquad &(492) \end{aligned}\right\} \text{(CHECK FORMULAE)}$$

$$y_c = \frac{(c - x_k)\, \zeta_{k+1}}{\varepsilon_{k+1}} + y_k \tag{493}$$

$$z_c = \frac{(c - x_k)\, \eta_{k+1}}{\varepsilon_{k-1}} + z_k \tag{494}$$

The initial constants are:

$$y_1 = 0 \text{ and } z_1 = {}_U h_1 \text{ and, } \epsilon_1 = n_0 \cos \alpha_0 \tag{495}$$

$$\zeta_1 = n_0 \sin \alpha_0 \tag{496}$$

$$\eta_1 = n_0 \cos 90^\circ = 0 \tag{497}$$

$$x_1 = -\sqrt{R_1^2 - z_1^2} \tag{498}$$

$$\cos i_1 = x_1\, \epsilon_1\, \rho_1 \tag{499}$$

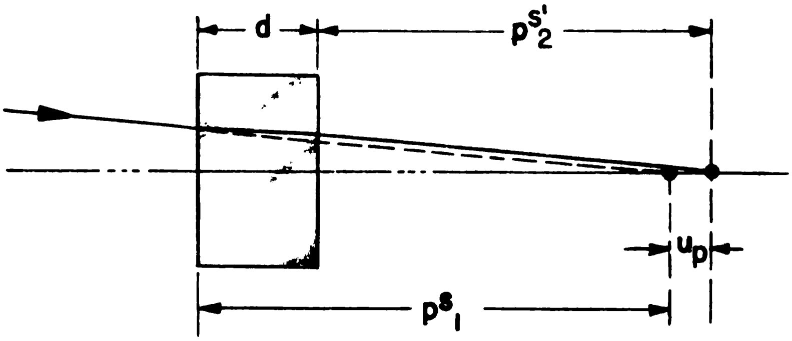


Figure 125.

The Paraxial Ray

${}_pS_1$ is given.

$$ {}_pS_v' = \frac{n_v\, {}_pS_v}{n_{v-1}} \tag{500}$$

$$ {}_pS_2 = {}_pS_1' - d \tag{501}$$

The axial displacement of the image plane caused by the plate is computed as follows:

$$u_p = {}_pS_2' + d - {}_pS_1 = \frac{(n-1)d}{n} \tag{502}$$

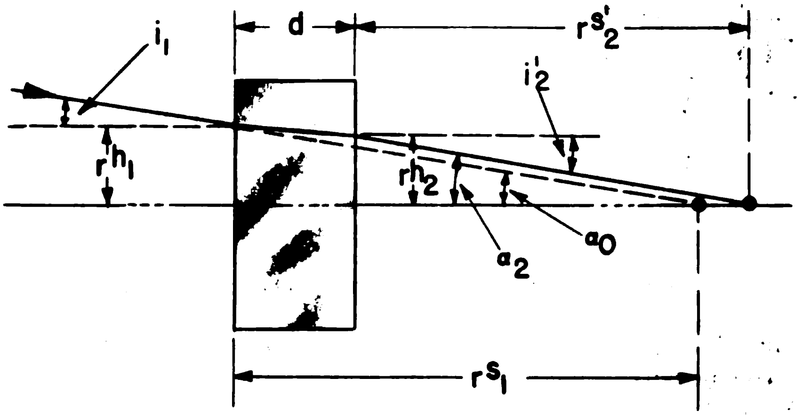


Figure 126.

The Rim Ray

${}_rs_1$ and α_0 are given.

$$i_1 = i_2' = -\alpha_0 = -\alpha_2 \qquad (503)$$

$$\sin i_1' = \frac{\sin i_1}{n} \qquad (504)$$

$$i_2 = -\alpha_1 = i_1 \qquad (505)$$

$${}_rs_v' = s_v \tan \alpha_{v-1} \operatorname{cotan} \alpha_v \qquad (506)$$

$${}_rs_{v+1} = {}_rs_v' - d_v \qquad (507)$$

$${}_rh_v = {}_rs_v \tan \alpha_{v-1} = {}_rs_v' \tan \alpha_v \qquad (508)$$

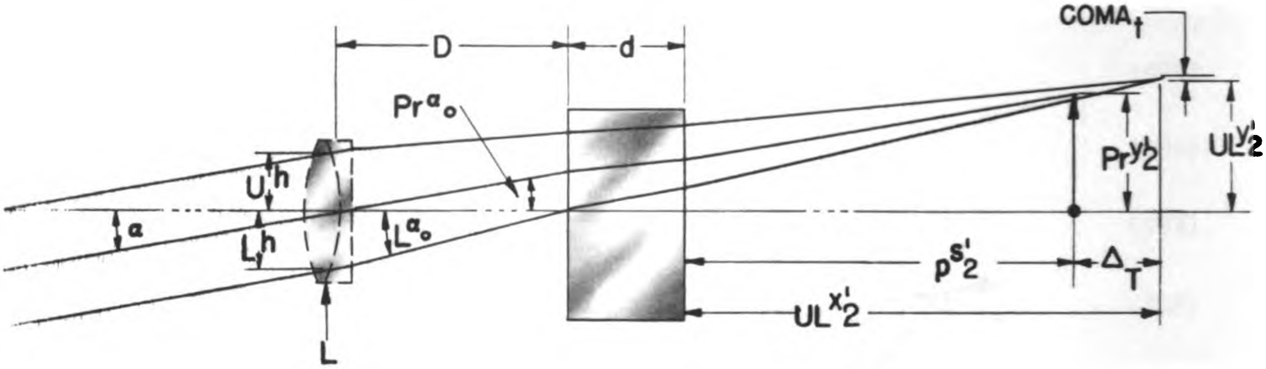


Figure 127.

Principal Ray—Upper and Lower Rim Rays

In the above sketch the objective L is shown placed at a distance D from a plane parallel plate (or prism). The upper and lower rim rays enter the objective at the distance $_U h$ above the axis and $_L h$ below the axis. f is the equivalent focal length of the objective. The initial constants are computed as follows:

$$_U s = \frac{_U h}{\tan \alpha} \qquad _{Pr}s = \pm 0 \qquad _L s = \frac{_L h}{\tan \alpha}$$

$$_U s' = \frac{f\,_U s}{f + {_U s}} \qquad _{Pr}s' = \pm 0 \qquad _L s' = \frac{f\,_L s}{f + {_L s}}$$

$$_U x_1 = {_U s_1'} - D \qquad _{Pr}x_1 = -D \qquad _L x_1 = {_L s_1'} - D$$

$$\tan {_U\alpha_o} = \frac{_U h}{_U s_1'} \qquad \tan {_{Pr}\alpha_o} \text{ is given} \qquad \tan {_L\alpha_o} = \frac{_L h}{_L s_1'}$$

Apply successively equations (503), (504), and (505) and then the equations following:

$$x_v' = x_v \tan \alpha_{v-1} \operatorname{cotan} \alpha_v \qquad (509)$$

$$x_2 = x_1' - d \qquad (510)$$

$$y_v = x_v \tan \alpha_{v-1} = x_v' \tan \alpha_v \qquad (511)$$

$$y' = -f \tan \alpha \qquad (449)$$

$$y_2' = (x_2' - {_p s_2'}) \tan \alpha_2 \qquad (512)$$

$$d' = \frac{d}{\cos \alpha_1} \qquad (513)$$

$$_{UL}x_2' = \frac{_U y_2 - {_L y_2}}{\tan {_U\alpha_2} - \tan {_L\alpha_2}} \qquad (514)$$

Now apply equation (453) (but substitute s_2' for ${}_s s_2'$) and then equation (454).

$$Q_2 = \frac{{}_p S_2'}{\cos \alpha_2} \tag{515}$$

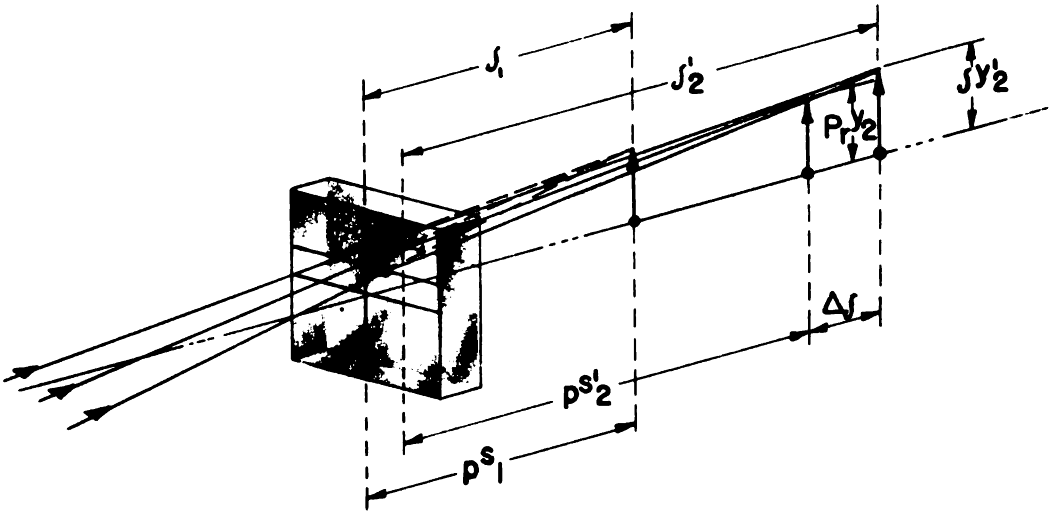


Figure 128.

The Sagittal Ray

Culmann's Formulae

NOTE: The values of α and Q_2 used in the tracing of the sagittal and the tangential are taken for the principal ray trace.

$$S_1 = \frac{{}_p S_1}{\cos \alpha_0} \tag{516}$$

$$S_2' = S_1 - \frac{d}{n_1 \cos \alpha_1} \tag{517}$$

$$\Delta S = (S_2' - Q_2) \cos \alpha_2 \tag{518}$$

$${}_s y_2' = x_2' \tan \alpha_2 - S_2' \sin \alpha_2 \tag{519}$$

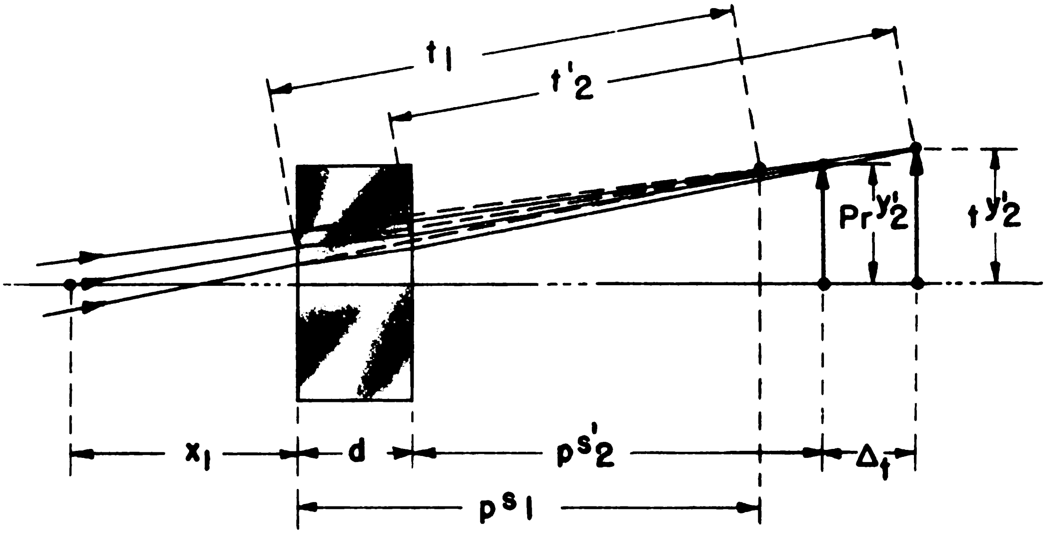


Figure 129.

The Tangential Ray

Culmann's Formulae

$$t_1 = S_1 \tag{520}$$

$$t_2' = t_1 - \frac{d \cos^2 \alpha_0}{n_1 \cos^3 \alpha_0} \tag{521}$$

$$\Delta t = (t_2' - Q_2) \cos \alpha_2 \tag{522}$$

$${}_t y_2' = x_2' \tan \alpha_2 - t_2' \sin \alpha_2 \tag{523}$$

90. Application of Formulae

Residual Aberrations

The ray tracing formulae for spherical surfaces will be applied to an equi-biconvex lens in air as shown in figure 130.

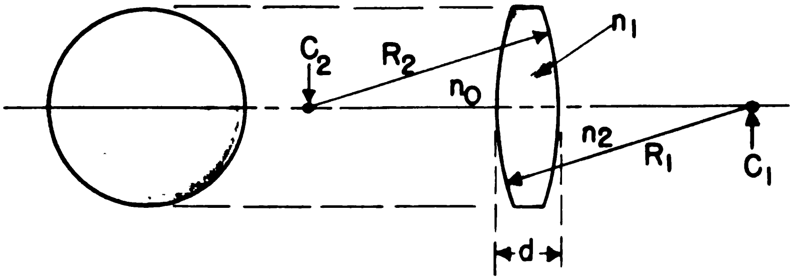


Figure 130.

The following constants of the lens are known:

$R_1 = +1.97897$

$R_2 = -1.97897$

$d = 0.50$

Since the lens is in air, $n_0 = n_2 = 1$

$n_1 = 1.5170$

The ray tracing formulae for plane surfaces will be applied to a plane-parallel plate in air (see fig. 131).

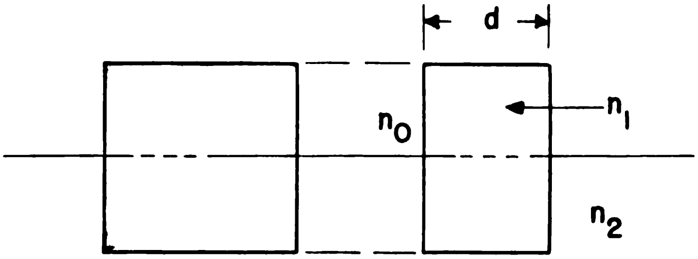


Figure 131.

The known constants of the plate are:

$d = 1.00$ $n_0 = n_2 = 1$ $n_1 = 1.5170$

Trace Through Lens

Object at Infinite Distance

The Zero Ray (${}_s s_1 = \infty$)

(See fig. 113, and apply equations (422) to (431).)

$v =$	1	2
$R_v =$	+ 1.9789657	— 1.9789657
$d_v =$	0.50	— — —
$n_v =$	1.5170	1.0000
$a_v =$	0	0.2858620
$+ b_v =$	0.1722132	0.2612475
$\sigma_v' =$	0.1722132	0.5471095
$d_v \sigma_v' =$	0.0861066	
$1 - d_v \sigma_v' =$	0.9138934	
$\sigma_{v+1} =$	0.1884390	
${}_s h_v =$	1.000	0.9138934

${}_s \alpha_2 = 0.50$

${}_s s_2' = 1.8277876$

${}_s f = 2.00$

$H = -H' = 0.1722124$

The Rim Ray (${}_r h_1 = +0.50$)

(See fig. 115, and apply equations (440), (432), (436), (438), (439), and (441).)

$v =$	1	2
$\sin i_v =$	0.2526572	— 0.3202845
$\sin i_v' =$	0.1665505	— 0.4858716
$i_v =$	14.6348	— 18.6801
$i_v' =$	9.5873	— 29.0696
$\alpha_v =$	5.0475	15.4370
$\sin \alpha_v =$	0.0879816	0.2661786
$s_v' - R_v =$	3.7462120	3.6123235
${}_r s_v' =$	5.7251777	
${}_r s_{v+1} =$	5.2251777	
$\omega_v =$	14.6348	— 13.6326
${}_r h_v =$	+ 0.50	+ 0.4664

${}_r s_2' = 1.6333578$

${}_r f = 1.8784380$

SA is the longitudinal spherical aberration and SC is the Seidel sum for the sine condition. Both constants will be used in computing the sine condition error as soon as the value of x_2' has been determined.

$$SA = {}_s s_2' - {}_r s_2' = 1.8277876 - 1.6333578 = +0.1944298 \text{ (under-corrected)}. \qquad (524)$$

$$SC = {}_s f - {}_r f = 2 - 1.878438 = +0.1215620 \qquad (525)$$

The Principal Ray ($x_1 = 0 \quad \alpha_0 = -5°$)

(See fig. 116, and apply equations (442) to (449).)

$v =$	1	2
$\sin i_v =$	0.0871557	0.0429367
$\sin i_v' =$	0.0574526	0.0651350
$i_v =$	+ 5.0000	+ 2.4609
$i_v' =$	+ 3.2936	+ 3.7346
$\alpha_v =$	— 3.2936	— 4.5673
$\sin \alpha_v =$	— 0.0574526	— 0.0796300
$x_v' - R_v =$	— 1.9789657	+ 1.6187357
$x_v' =$	± 0	— 0.3602300
$x_{v+1} =$	— 0.50	
$y_v' =$		+ 0.1747869
$\omega_v =$	± 0	— 0.8327
$y_v =$	± 0	+ 0.0287601
$d_v' =$	0.5005883	

DISTORTION $= y_2' - y'$ (526)

DIST $= -0.0001905$

(Barrel Shape)

The Upper and the Lower Rim Rays

(See figs. 118 and 119, and then apply the equations associated with them.)

RAY	UPPER		LOWER	
$v =$	1	2	1	2
$h_1 =$	+ 0.50		— 0.50	
$\sin \varepsilon =$	0.2526572		— 0.2526572	
$\varepsilon =$	14.6348		— 14.6348	
$i_1 =$	19.6348		— 9.6348	
$\sin i_v =$	0.3360237	— 0.2775278	— 0.1673676	0.3625859
$\sin i_v' =$	0.2215054	— 0.4210097	— 0.1103280	0.5500428
$i_v =$	19.6348	— 16.1127	— 9.6348	21.2591
$i_v' =$	12.7975	— 24.8984	— 6.3342	33.3700
$\alpha_v =$	1.8373	+ 10.6230	— 8.3006	— 20.4115
$\sin \alpha_v =$	0.0320615	0.1843459	— 0.1443666	— 0.3487602
$x_v' - R_v =$	+ 13.6722108	+ 4.5195675	+ 1.5123673	+ 3.1211010
$x_v' =$	+ 15.6511765	+ 2.5406018	+ 3.4913330	+ 1.1421353
$x_{v+1} =$	+ 15.1511765		+ 2.9913330	
$y_v' =$		+ 0.1336957		+ 0.2551485
$\omega_v =$	+ 14.6348	— 14.2754	— 14.6348	+ 12.9585
$y_v =$	+ 0.50	+ 0.4879793	— 0.50	— 0.4437735
$d_v' =$	0.3749263		0.3894702	

Applying equations (452), (453), and (454) we find:

$${}_{UL}x_2' = +1.6107857 \qquad \Delta T = -0.2170019 \qquad {}_{UL}y_2' = +0.1743966$$

The sine condition error, or the offence against the sine condition is computed as follows:

$$(SCE) = \frac{SC}{{}_{s}f} - \frac{SA}{{}_{s}s_2' - {}_{Pr}x_2'} = -0.0280801 \text{ (approximate value)} \tag{527}$$

$$(SCE) = 1 - \frac{{}_{r}f\,({}_{s}s_2' - {}_{Pr}x_2')}{{}_{s}f\,({}_{r}s_2' - {}_{Pr}x_2')} = -0.0308187 \text{ (correct value)} \tag{528}$$

The amount of coma in the tangential and the sagittal planes is computed as follows:

$$\text{Coma}_T = ({}_{Pr}x_2' - {}_{UL}x_2') \tan {}_{Pr}\alpha_2 - {}_{UL}y_2' = {}_{Pr}y_2' - {}_{UL}y_2' - \Delta T \tan {}_{Pr}\alpha_2 = 0.01695 \tag{529}$$

$$\text{Coma}_S = \frac{\text{Coma}_T}{3} = -0.00565 \tag{530}$$

We compute the values of A_S and A_T as follows:

$$A_S = \frac{\text{Coma}_S - (SCE)y'}{(y')^3} = -0.0480510 \tag{531}$$

$$A_T = \frac{\text{Coma}_T - 3(SCE)y'}{(y')^3} = -0.1441532 \tag{532}$$

Equations (529) and (530) may now be written as follows:

$$\text{Coma}_T = 3(SCE)y' + A_T(y')^3 = -0.01695 \tag{533}$$

$$\text{Coma}_S = (SCE)y' + A_S(y')^3 = -0.00565 \tag{534}$$

The Sagittal Ray ($S_1 = \infty$)

(See fig. 120, and apply equations (456) to (463).)

$v =$	1	2
${}_{S}a_v =$ $+\,{}_{S}b_v =$	0 0.1726461	0.2866802 0.2616137
${}_{S}\sigma_v' =$ $d_v'\,{}_{S}\sigma_v' =$ $1 - d_v'\,{}_{S}\sigma_v' =$	0.1726461 0.0864246 0.9135754	0.5482939
${}_{S}\sigma_{v+1} =$	0.1889784	

$T_2 = -0.0002090$

$S_2' = +1.8238393$

$\Delta S = -0.0099491$

The Tangential Ray ($t_1 = \infty$)

(See fig. 122, and apply equations (467) to (473).)

$v =$	1	2
${}_t a_v =$	± 0	0.2884129
$+ {}_t b_v =$	$+ 0.1732178$	0.2627283
${}_t\sigma_v' =$	$+ 0.1732178$	0.5511412
$d_v' {}_t\sigma_v' =$	0.0867108	
$1 - d_v' {}_t\sigma_v' =$	0.9132892	
${}_t\sigma_{v+1} =$	0.1896636	

$T_2 = - 0.0002090$

$t_2' = + 1.8144170$

$\Delta t = - 0.0193415$

$$\text{ASTIGMATISM} = \text{AST} = \Delta t - \Delta S = - 0.0093924 \tag{535}$$

$$\text{MEAN FIELD CURVATURE} = \text{MFC} = \frac{\Delta S + \Delta t}{2} = - 0.0146453 \tag{536}$$

$$M_v = \frac{n_v - n_{v-1}}{n_{v-1}\, n_v} \qquad M_1 = - M_2 = + 0.3408042 \tag{537}$$

$$\text{PETZVAL SAGITTA} = \text{PS} = \frac{(y')^2}{2} \sum_1^2 \frac{M_v}{R_v} = - 0.0052727 \tag{538}$$

$$\text{AST}_T = \Delta t - \text{PS} = - 0.0140688 \tag{539}$$

$$\text{AST}_S = \Delta S - \text{PS} = - 0.0046764 \tag{540}$$

$$\frac{\text{AST}_T}{\text{AST}_S} = 3.00846 \quad \text{(This quotient will be equal to 3 in the absence of coma.)} \tag{541}$$

The Skew Ray

(See fig. 124.) Determine the initial constants with equations (495) to (499) and then apply equations (477) to (494).

$\alpha_0 = -50$ $y_1 = \pm 0$ $z_1 = +0.50$

$\varepsilon_1 = +0.9961947$ $\zeta_1 = +0.0871557$ $\eta_1 = \pm 0$

$x_1 = -1.9147598$ $\cos i_1 = -0.9638739$ $i_1 = 15.4477$

$v =$	1	2
$\sin i_v =$	0.2663587	0.3234757
$\sin i_v' =$	0.1755825	0.4907126
$i_v =$	15.4477	18.8733
$i_v' =$	10.1126	29.3874
$n_v \cos i_v' =$	− 1.4934328	+ 0.8713218
$-n_{v-1} \cos i_v =$	+ 0.9638739	− 1.4354403
$\tau_v =$	− 0.5295589	− 0.5641185
$\varepsilon_v =$	+ 0.9961947	+ 1.5085725
$+X_v \tau_v \rho_v =$	+ 0.5123778	− 0.5481988
$\varepsilon_{v+1} =$	+ 1.5085725	− 0.9603737
$\zeta_v =$	+ 0.0871557	+ 0.0871557
$+y_v \tau_v \rho_v =$	± 0	− 0.0062572
$\zeta_{v+1} =$	+ 0.0871557	+ 0.0808985
$\eta_v =$	± 0	− 0.1337968
$+z_v \tau_v \rho_v =$	− 0.1337968	− 0.1329227
$\eta_{v+1} =$	− 0.1337968	− 0.2667195
$U_v =$	+ 1.6221525	
$\varepsilon_{v+1} K_{v+1} =$	+ 5.2165402	
$+n_v R_v \cos i_v' =$	− 2.9554523	
$\cos \beta_v =$	0.9188406	
$\beta_v =$	23.2428	
$R_{v+1} \cos i_{v+1} =$	+ 1.8725690	
$-U_1 \cos \beta_v =$	− 1.4904996	
$V_{v+1} =$	+ 0.2518585	

$x_2 = +1.9231184$

$y_2 = +0.0219509$

$z_2 = +0.4663021$

$\varepsilon = +3.8067533$

$v =$	1	2	3
$\varepsilon_v^2 =$	0.9924039	2.2757910	0.9223176
$+ \zeta_v^2 =$	0	0.0075961	0.0065446
$+ \eta_v^2 =$	0.0075961	0.0179016	0.0711393
$=$	1.0000	2.3012887	1.0000
$n^2_{v-1} =$	1.0000	2.3012890	1.0000
$x_v^2 =$	3.6663051	3.6983844	
$+ y_v^2 =$	0	0.0004818	
$+ z_v^2 =$	0.25	0.2174376	
$=$	3.9163051	3.9163038	
$R_v^2 =$	3.9163052	3.9163052	
$x_v \varepsilon_v \rho_v =$	$-$ 0.9638739	$+$ 1.4659999	
$+ y_v \zeta_v \rho_v =$	0	$+$ 0.0009667	
$+ z_v \eta_v \rho_v =$	0	$-$ 0.0315264	
$=$	$-$ 0.9638739	$+$ 1.4354402	
$n_{v-1} \cos i_v =$	$-$ 0.9638739	$+$ 1.4354403	

CHECK

FORMULAE

$y_c = -0.1806215$ $z_c = -0.0568298$

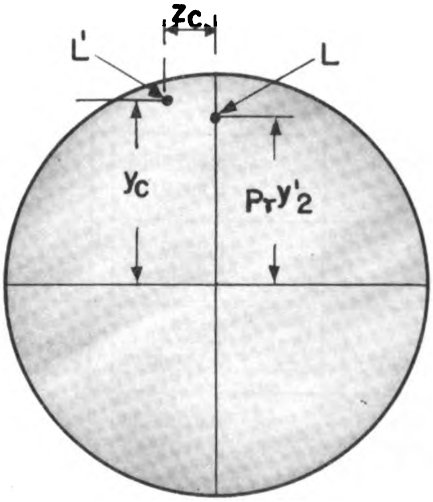


Figure 132.

The Image Plane

Due to the presence of zonal errors in the field curvature the image of a point-object will be elongated in the shape of a wing. Its maximum length will be equal to

$$(L - L') = \sqrt{({}_{Pr}y_2' - y_c)^2 + (z_c)^2} = 0.0571279 \qquad (542)$$

Object at Finite Distance

The Paraxial Ray ($_ps_1 = -3.8277877$)

(See fig. 114, and apply associated equations.)

$v =$	1	2
$R_v =$	+ 1.9789657	— 1.9789657
$\rho_v =$	+ 0.5053144	— 0.5053144
$d_v =$	0.50	————
$n_v =$	1.5170	1.0000
$a_v =$	— 0.1722132	± 0
$+ b_v =$	+ 0.1722132	+ 0.2612475
$\sigma_v' =$	± 0	+ 0.2612475
$d_v \sigma_v' =$	± 0	
$1 - d_v \sigma_v' =$	1.0000	
$\sigma_{v+1} =$	± 0	

$$_ps_2' = +3.8277878$$

Assume a value for $_ph_1$, say

$_ph_1 = +0.50$, then

$$\tan {_p\alpha_0} = \frac{_ph_1}{_ps_1} = -0.1306237$$

$$_ph_2 = (1 - d_1 \sigma_1')\, {_ph_1} = +0.50$$

$$\tan {_p\alpha_2} = \frac{_ph_2}{_ps_2'} = 0.1306237$$

$$_p\alpha_0 = -{_p\alpha_2} = -7.4421^\circ$$

The Rim Ray ($_rs_1 = -3.8277877$ $\quad _rh_1 = +0.50$)

(See fig. 117, and apply associated equations.)

$$\tan {_r\alpha_0} = \frac{_rh_1}{_rs_1} = -0.1306237 \qquad _r\alpha_0 = -7.4421^\circ$$

$v =$	1	2
$\sin i_v =$	0.3800546	— 0.2623051
$\sin i_v' =$	0.2505303	— 0.3979168
$i_v =$	22.3371	— 15.2069
$i_v' =$	14.5089	— 23.4480
$\alpha_v =$	0.3861	+ 8.6272
$\sin \alpha_v =$	0.0067387	0.1500047
$s_v' - R_v =$	73.5736671	5.2495934
$s_v' =$	75.5526328	3.2706277
$s_{v+1} =$	75.0526328	
$\omega_v =$	14.8950	— 14.8208
$h_v =$	+ 0.5087	+ 0.5062

$$_rs_2' = +3.2706277$$

$$SA = {_ps_2'} - {_rs_2'} = +0.557160$$

$$x_2' = -0.3602300$$

(see Principal Ray.)

$$(SCE) = 1 - \frac{_ph_1\, {_rs_1} \sin {_p\alpha_2} ({_ps_2'} - x_2')}{_rh_1\, {_ps_1} \sin {_r\alpha_2} ({_rs_2'} - x_2')} = +0.004034 \qquad (543)$$

The Principal Ray

$({}_{p}s_1 = -3.8277877 \quad x_1 = 0 \quad {}_{U}h_1 = {}_{L}h_1 = +0.50 \quad \alpha_0 = -5°)$

(See fig. 117, where the diaphragm is shown at the distance x_1 from the first surface. Apply equations (442), (443), (433), (434), (444), (445), (438), (446), (448), (449), (450), and (451).)

$v =$	1	2
$\sin i_v =$	0.0871557	0.0429367
$\sin i_v' =$	0.0574526	0.0651350
$i_v =$	+ 5.0000	+ 2.4609
$i_v' =$	+ 3.2936	+ 3.7346
$\alpha_v =$	— 3.2936	— 4.5673
$\sin \alpha_v =$	— 0.0574526	0.0796300
$x_v' - R_v =$	— 1.9789659	+ 1.6187357
$x_v' =$	± 0	
$x_{v+1} =$	— 0.50	
$\omega_v =$	± 0	— 0.8327
$y_v =$	± 0	+ 0.0287601
$d_v' =$	0.5005883	

$y' = +0.3348882$

$x_2' = -0.3602300$

$y_2' = +0.3345544$

DIST = — 0.0003338

(Barrel Shape)

The Upper and Lower Rim Rays

(See fig. 118. Determine the initial constants with the equations associated with figure 119, and then apply equations (443), (433), (434), (445), (438), (440), (441), (450), and (451).)

RAY =	UPPER		LOWER	
$v =$	1	2	1	2
$y =$	— 0.3348882		— 0.3348882	
$\tan \alpha_0 =$	— 0.2181124		+ 0.0431350	
$\alpha_0 =$	— 12.3042		+ 2.4699	
$\varepsilon =$	+ 14.6348		— 14.6348	
$i_1 =$	+ 26.9390		— 17.1047	
$x_1 =$	— 2.2923960		— 11.5915150	
$\sin i_v =$	0.4530416	— 0.2150738	— 0.2941187	0.2991969
$\sin i_v' =$	0.2986431	— 0.3262670	— 0.1938818	0.4538817
$i_v =$	+ 26.9390	— 12.4199	— 17.1047	+ 17.4094
$i_v' =$	+ 17.3761	— 19.0424	— 11.1794	+ 26.9930
$\alpha_v =$	— 2.7413	+ 3.8812	— 3.4554	— 13.0390
$\sin \alpha_v =$	— 0.0478265	0.0676879	— 0.0602716	— 0.2256142
$x_v' - R_v =$	— 12.3572590	+ 9.5389456	+ 6.3659407	+ 3.9812047
$x_v' =$	— 10.3782933	+ 7.5599799	+ 8.3449064	+ 2.0022390
$x_{v+1} =$	— 10.8782933		+ 7.8449064	
$\omega_v =$	+ 14.6348	— 15.1612	— 14.6348	+ 13.9540
$y_v =$	+ 0.50	+ 0.5175702	— 0.50	— 0.4772133
$d_v' =$	0.3673737		0.3780669	

From the preceding computations we find:

$_{UL}x_2' = + 3.2614918$

$_{UL}y_2' = - 0.2916245$

$\Delta T = - 0.5662960$

$\text{Coma}_T = - 0.0023079$

$\text{Coma}_S = - 0.0007693$

The Sagittal Ray

$({}_{p}s_1 = - 3.8277877 \quad x_1 = 0 \quad \alpha_0 = - 5°)$

(See fig. 121, and apply equations (465), (457), (458), (459), (460), (461), (462), and (466).)

$v =$	1	2
$_{s}a_v =$	$- 0.1715579$	$+ 0.0016516$
$+_{s}b_v =$	$+ 0.1726461$	$+ 0.2616137$
$_{s}\sigma_v' =$	$+ 0.0010882$	$+ 0.2632653$
$d_v'\ {}_{s}\sigma_v' =$	$+ 0.0005447$	
$1 - d_v'\ {}_{s}\sigma_v' =$	$+ 0.9994553$	
$_{s}\sigma_{v+1} =$	$+ 0.0010887$	

$T_2 = - 0.0002090$

$S_2' = + 3.7984497$

$\Delta_S = - 0.0416094$

The Tangential Ray

$({}_{p}s_1 = - 3.8277877 \quad x_1 = 0 \quad \alpha_0 = - 5°)$

(See fig. 123, and apply equations (495), (468) to (472), and (476).)

$v =$	1	2
$_{t}a_v =$	$- 0.1708186$	$+ 0.0036526$
$+_{t}b_v =$	$+ 0.1732178$	$+ 0.2627283$
$_{t}\sigma_v' =$	$+ 0.0023992$	$+ 0.2663809$
$d_v' + \sigma_v' =$	$+ 0.0012010$	
$1 - d_v' + \sigma_v' =$	$+ 0.9987990$	
$t_{\sigma_{v+1}} =$	$+ 0.0024020$	

$T_2 = - 0.0002090$

$t_2' = + 3.7540229$

$\Delta_t = - 0.0858951$

With the aid of equations (535) to (540) we find:

$AST = - 0.0442857$

$AST_S = - 0.0222957$

$PS = - 0.0193137$

$MFC = - 0.0637522$

$AST_T = - 0.0665814$

Trace Through Plane Parallel Plate

The Paraxial Ray

(See fig. 125, and apply equations (500) to (502).)

$d = 1.00$ $\quad {}_p s_1 = +4.000$ $\quad {}_p s_1' = +6.068$

${}_p s_2 = +5.068$ $\quad {}_p s_2' = +3.3408$ $\quad u_p = +0.3408$

The Rim Ray

(See fig. 126, and apply equations (503) to (508).)

${}_r s_1 = +4.000$ $\quad \alpha_0 = 10°$

$v =$	1	2
$\sin i_v =$	-0.1736482	-0.1144681
$\sin i_v' =$	-0.1144681	-0.1736482
$i_v =$	-10.0000	-6.5729
$i_v' =$	-6.5729	-10.0000
$\alpha_v =$	$+6.5729$	$+10.0000$
$\tan \alpha_v =$	0.1152246	0.1763270
${}_r s_v' =$	$+6.1211581$	
${}_r s_{v+1} =$	$+5.1211581$	
${}_r h_v =$	$+0.7053080$	$+0.5900834$

${}_r s_2' = +3.3465$

SA $= -0.0057$

The Principal Ray

(See fig. 127, and apply the equations associated with it.)

EFL of objective = 6′00″

${}_p s_1 = +4.00$ $\quad x_1 = -2.00$ $\quad \alpha_0 = -10°$

$v =$	1	2
$\tan \alpha_{v-1} =$	-0.1763270	-0.1152246
$\sin \alpha_{v-1} =$	-0.1736482	-0.1144681
$\sin \alpha_v =$	-0.1144681	-0.1736481
$\alpha_v =$	-6.5729	-10.0000
$\tan \alpha_v =$	-0.1152246	-0.1763270
$x_v' =$	-3.0605790	-2.6534710
$x_{v+1} =$	-4.0605790	
$y_v =$	$+0.3526540$	$+0.4678786$

$y' = 1.0579620$

$y_2' = 1.0569518$

$d' = 1.0066164$

$Q_2 = 3.3923370$

DIST $= -0.0010094$ (Barrel Shape)

The Upper and Lower Rim Rays

(See fig. 127, and apply the equations associated with it.)

RAY =	UPPER		LOWER	
v =	1	2	1	2
$x_v =$	— 7.3767098		— 0.0744076	
tan $\alpha_{v-1} =$	— 0.0929936		— 0.2596603	
sin $\alpha_{v-1} =$	— 0.0925948	— 0.0610381	— 0.2513261	— 0.1656731
sin $\alpha_v =$	— 0.0610381	— 0.0925948	— 0.1656731	— 0.2513261
$\alpha_v =$	— 3.4994	— 5.3129	— 9.5363	— 14.5560
tan $\alpha_v =$	— 0.0611521	— 0.0929936	— 0.1679940	— 0.2596603
$x_v' =$	— 11.2177145	— 8.0342892	— 0.1150082	— 0.7213886
$x_{v+1} =$	— 12.2177145		— 1.1150082	
$y_v =$	+ 0.6859868	+ 0.7471369	+ 0.0193207	+ 0.1873147
$y_k' =$		+ 1.057813		+ 1.0547878

$_{UL}x_2' = +3.3589349$ $\qquad$ $\text{Coma}_T = +0.0006536$

$\Delta T = +0.0181349$

$_{UL}y_2' = +1.0594967$ $\qquad$ $\text{Coma}_S = +0.0002176$

The Sagittal Ray

(See fig. 128, and apply equations (516) to (519).)

$_pS_1 = +4.00$ $\qquad$ $S_2' = +3.3981489$

$S_1 = +4.0617062$

$\Delta_S = +0.0057742$ $\qquad$ $_Sy_2' = +1.057961$

The Tangential Ray

(See fig. 129, and apply equations (520) to (526).)

$_pS_1 = +4.00$ $\qquad$ $t_1 = +4.0617062$

$t_2' = +3.4096141$ $\qquad$ $\Delta t = +0.0170146$

$_ty_2' = +1.0599518$

AST $= +0.0112404$ $\qquad$ MFC $= +0.0113944$

SECTION VII

ACHROMATIC LENSES

91. Introduction

In the last two sections it has been tacitly assumed that the index of the glass is constant. However, from paragraph 6, we know that the refractive index of the glass depends on the wave-length λ of the light and, consequently, the images formed by red, yellow, and blue light are located in different planes and are also of different sizes. This phenomena is known as the *"CHROMATIC ABERRATION"* of the lens. If the images for two colors, say for the red and the blue light, lie in one plane, we say the lens is *"ACHROMATIC,"* and if these images are also of the same size we speak of *"STABLE ACHROMATISM."* In the following paragraphs, constants having reference to the red, yellow, or blue light will carry the subscripts C, D, or F at the lower left.

CA = the longitudinal chromatic aberration, that is, ${}_Cs_k' - {}_Fs_k'$

LCA = the lateral chromatic aberration, that is, ${}_Cy' - {}_Fy'$

92. Thin Lens

The focal length of the thin lens is computed with equation (206), thus,

$$f = \frac{R_1R_2}{(n-1)(R_2-R_1)}$$

If we substitute the values of ${}_Cn$, ${}_Dn$ and ${}_Fn$, respectively, we find the values of ${}_Cf$, ${}_Df$, and ${}_Ff$, respectively. In the case of a thin lens $s_2' = f$, hence the longitudinal chromatic aberration is

$$CA = {}_Cf - {}_Ff \tag{544}$$

If the principal ray includes the angle α with the axis then

$$y' = f \tan \alpha \tag{278}$$

The lateral chromatic aberration is equal to

$$LCA = {}_Cy' - {}_Fy' \tag{545}$$

Example 1:

The collective lens shown in figure 133(a) and (b) has the following constants:

$R_1 = +1.7051$ $\quad R_2 = -1.8795$ $\quad d = \pm 0$ and $\alpha = -10°$

The refractive indices of the glass are as follows:

${}_Cn = 1.51462$ $\quad {}_Dn = 1.5170$ and ${}_Fn = 1.52264$

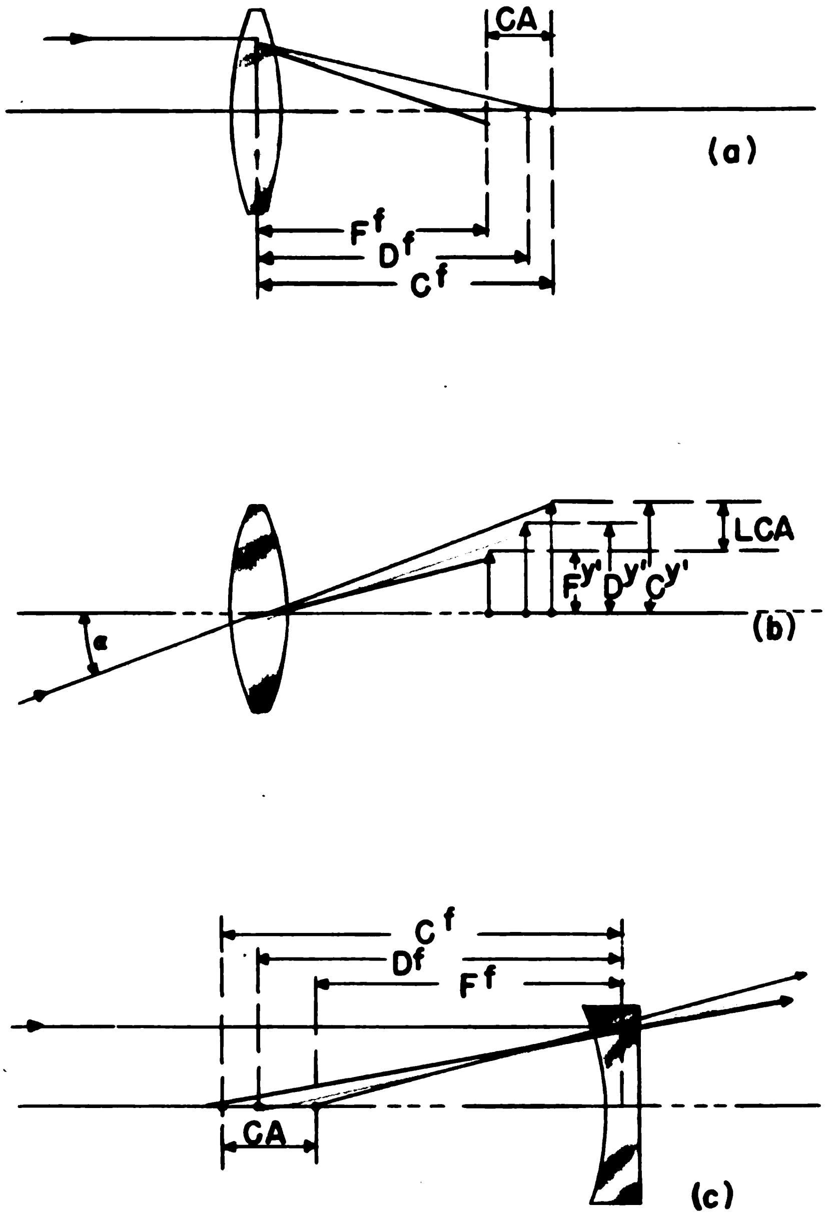


Figure 133.

Applying equations (206) and (278) we find,

$$_Cf = \frac{R_1R_2}{(_Cn - 1)(R_2 - R_1)} = 1.7372 \qquad _Cy' = -\,_Cf \tan\alpha = 0.3063$$

$$_Df = \frac{R_1R_2}{(_Dn - 1)(R_2 - R_1)} = 1.7292 \qquad _Dy' = -\,_Df \tan\alpha = 0.3049$$

$$_Ff = \frac{R_1R_2}{(_Fn - 1)(R_2 - R_1)} = 1.7106 \qquad _Fy' = -\,_Ff \tan\alpha = 0.3016$$

Substituting these values into equations (544) and (545) we find:

$$CA = 1.7372 - 1.7106 = +0.0266 \qquad LCA = 0.3063 - 0.3016 = +0.0047$$

Both values being positive, we say the lens is chromatically *"UNDER-CORRECTED."* The terms $_Cf - {_Df}$ or $_Df - {_Ff}$ are called the *"SECONDARY CHROMATIC ABERRATION"* of the lens.

Example 2:

In figure 133(c) is shown a dispersive lens which has the following constants:

$$R_1 = -1.8795 \qquad R_2 = \pm\infty \qquad \text{and} \qquad d = \pm 0 \qquad \alpha = -10^\circ$$

The refractive indices of the glass are:

$$_Cn = 1.61217 \qquad _Dn = 1.6170 \qquad _Fn = 1.62903$$

Substituting these values into equations (232) and (278) we find,

$$_Cf = \frac{R_1}{_Cn - 1} = -3.0701 \qquad _Cy' = -0.5413$$

$$_Df = \frac{R_1}{_Dn - 1} = -3.0461 \qquad _Dy' = -0.5371 \qquad CA = -0.0822$$

$$_Ff = \frac{R_1}{_Fn - 1} = -2.9879 \qquad _Fy' = -0.5268 \qquad LCA = -0.0145$$

Here the values of CA and LCA are negative and this lens is chromatically *"OVER-CORRECTED."*

The facts that the collective lens is under-corrected, and the dispersive lens is over-corrected suggest a method of obtaining an achromatic doublet, namely, by combining a collective lens with a dispersive lens of suitable focal length. This method will be explained in the following paragraphs.

93. Thin Achromatic Doublet

In the following, formulae will be given for computing the radii of lens so that the system is achromatic, but no attention will be paid to the Seidel errors. Such lenses may be used in eye pieces or as magnifiers. In this paragraph, all constants have reference to light of the D–line, hence, the subscript D is omitted.

From equation (237), we have $\phi = \phi_1 + \phi_2$. This equation must be fulfilled if the thin doublet is to have a definite focal length and, if the doublet is to be achromatic, then the following equation must be satisfied.

$$\frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} = 0 \qquad \text{where: } \nu_1 = \frac{{}_Dn_1 - 1}{{}_Fn_1 - {}_Cn_1} = \frac{{}_Dn_1 - 1}{\Delta n_1} \qquad \nu_2 = \frac{{}_Dn_2 - 1}{{}_Fn_2 - {}_Cn_2} = \frac{{}_Dn_2 - 1}{\Delta n_2} \tag{546}$$

In some cases a certain amount of chromatic aberration is desired so that

$$\Gamma = {}_C\phi - {}_F\phi \tag{547}$$

then equation (546) will read,

$$\frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} = \Gamma \tag{548}$$

Equation (239) may be written as follows:

$$(n_1 - 1)(\rho_1 - \rho_2) + (n_2 - 1)(\rho_2 - \rho_3) = \phi \tag{549}$$

Equation (546) can be written

$$\Delta n_1 (\rho_1 - \rho_2) + \Delta n_2 (\rho_2 - \rho_3) = \Gamma \tag{550}$$

Two equations must be solved and we have three unknown values, namely, the three radii. Therefore, we may assume a value for one radius, or we specify that the radii are in definite ratio r to each other. We derive the equations for ρ_ν, assuming that $R_3 = \infty$, or $\rho_3 = 0$, as follows.

From equation (549),

$$(n_1 - 1)\rho_1 + (n_2 - n_1)\rho_2 = \phi \tag{551}$$

hence

$$\rho_1 = \frac{\phi - (n_2 - n_1)\rho_2}{n_1 - 1} \tag{552}$$

From equation (550),

$$\Delta n_1 \rho_1 + (\Delta n_2 - \Delta n_1)\rho_2 = \Gamma \tag{553}$$

Substitute equation (552) into this equation and we have,

$$\rho_2 = \frac{\Delta n_1 \phi - (n_1 - 1)\Gamma}{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)} \tag{554}$$

Substitute equation (554) into (552) and obtain,

$$\rho_1 = \frac{(n_2 - n_1)\Gamma - (\Delta n_2 - \Delta n_1)\phi}{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)} \tag{555}$$

If we consider the reciprocal values of ρ and ϕ we obtain the following set of equations:

$$R_1 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f}{(n_2 - n_1) f\Gamma - (\Delta n_2 - \Delta n_1)} \tag{556}$$

$$R_2 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f}{\Delta n_1 - (n_1 - 1) f\Gamma} \tag{557}$$

$$R_3 = \pm \infty \text{ (assumed)}$$

By a similar procedure we derive equations for the values of R by assuming that (a) R_2 is given, (b) $R_2 = -R_1$, (c) $R_3 = -R_1$, and (d) $R_3 = rR_1$.

94. Thin Achromatic Triplet

From equation (280) we have, $\phi = \phi_1 + \phi_2 + \phi_3$ and

$$\Gamma = \frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} + \frac{\phi_3}{\nu_3} \tag{558}$$

In the case of a triplet we may impose two conditions, for instance, (a) $R_1 = \pm \infty$, and $R_2 = -R_3$, or (b) $R_1 = -R_4$, and $R_2 = -R_3$. The formulae are derived by the same method as applied in the case of the doublet.

Example 3:

Determine the radii for a thin cemented achromatic doublet whose rear surface is plane.

The given constants are:

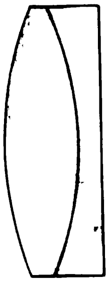
$f = 4.00$ inches $R_3 = \pm \infty$ and $\Gamma = \pm 0$

The characteristics of the glasses to be used are as follows:

LENS	Δn	$_Cn$	$_Dn$	$_Fn$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6

$$R_1 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f}{(n_2 - n_1) f\Gamma - (\Delta n_2 - \Delta n_1)} = +1.7051$$

$$R_2 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f}{\Delta n_1 - (n_1 - 1) f\Gamma} = -1.8795$$



$R_1 = +1.7051$

$R_2 = -1.8795$

$R_3 = \pm \infty$

$d_1 = \pm 0$

$d_2 = \pm 0$

Figure 134.

Trace zero rays through the system for light corresponding to the C, D, and F lines of the spectrum. The result is tabulated below.

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 4:

Determine the radii of a thin cemented achromatic doublet of which the second radius is known.

The given constants are:

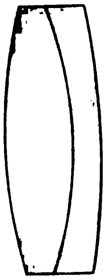
$f = 4.00$ inches $R_2 = -1.50$ inches and $\Gamma = \pm 0$

The characteristics of the glasses are as follows:

LENS	Δn	$_Cn$	$_Dn$	$_Fn$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6

$$R_1 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f R_2}{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f - \{\Delta n_2 - (n_2 - 1) f\Gamma\} R_2} = +2.2130$$

$$R_3 = \frac{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f R_2}{\{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\} f - \{\Delta n_1 - (n_1 - 1) f\Gamma\} R_2} = -7.4296$$



$R_1 = +2.2130$

$R_2 = -1.5000$

$R_3 = -7.4296$

$d_1 = \pm 0$

$d_2 = \pm 0$

Figure 135.

From the zero ray trace we find the values of f as follows:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 5:

Determine the radii of a thin cemented achromatic doublet so that the collective element will be symmetrical.

The given constants are:

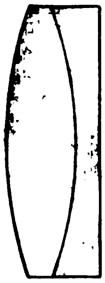
$$f = 4.00 \text{ inches} \qquad R_1 = -R_2 \qquad \text{and} \qquad \Gamma = \pm 0$$

The characteristics of the glasses to be used are as follows:

LENS	Δn	${}_C n$	${}_D n$	${}_F n$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6

$$R_1 = -R_2 = \frac{2f\ \{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\}}{(n_2 - 1)\ f\Gamma - \Delta n_2} = +1.7880$$

$$R_3 = \frac{2f\ \{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\}}{(2n_1 - n_2 - 1)\ f\Gamma - (2\Delta n_1 - \Delta n_2)} = -36.7639$$



$R_1 = +1.7880$

$R_2 = -1.7880$

$R_3 = -36.7639$

$d_1 = \pm 0$

$d_2 = \pm 0$

Figure 136.

From the zero ray trace we find the following values for f:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 6:

Compute the radii of a thin cemented achromatic doublet so that the absolute values of the front and the rear surface are identical.

The given constants are:

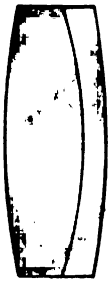
$f = 4.00$ inches $\qquad R_1 = -R_3 \qquad$ and $\qquad \Gamma = \pm 0$

The characteristics of the glasses to be used are as follows:

LENS	Δn	$_C n$	$_D n$	$_F n$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6

$$R_1 = -R_3 = \frac{2f\ \{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\}}{(n_2 - n_1) f\Gamma - (\Delta n_2 - \Delta n_1)} = +3.4102$$

$$R_2 = \frac{2f\ \{\Delta n_1 (n_2 - 1) - \Delta n_2 (n_1 - 1)\}}{(\Delta n_1 + \Delta n_2) - (n_1 + n_2 - 2) f\Gamma} = -1.2117$$



$R_1 = +3.4102$

$R_2 = -1.2117$

$R_3 = -3.4102$

$d_1 = \pm 0$

$d_2 = \pm 0$

Figure 137.

Tracing zero rays through the system we find:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 7:

Determine the radii of a thin cemented achromatic doublet so that the first and last radii have a definite ratio to each other.

We have:

$f = 4.00$ inches $\qquad R_3 = rR_1 \qquad r = -2 \qquad$ and $\qquad \Gamma = \pm 0$

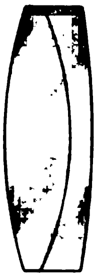
Glasses having the following characteristics will be used:

LENS	Δn	$_{C}n$	$_{D}n$	$_{F}n$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6

$$R_1 = \frac{\{(\Delta n_2 - \Delta n_1)\,[(n_1 - 1)\,r - (n_2 - 1)] - (n_2 - n_1)\,(\Delta n_1 r - \Delta n_2)\}\,f}{\{(\Delta n_2 - \Delta n_1) - (n_2 - n_1) f\Gamma\}\,r} = +\,2.5576$$

$$R_2 = \frac{\{(\Delta n_2 - \Delta n_1)\,[(n_1 - 1)\,r - (n_2 - 1)] - (n_2 - n_1)\,(\Delta n_1 r - \Delta n_2)\}\,f}{\{(n_1 - 1)\,r - (n_2 - 1)\}\,f\Gamma - (\Delta n_1 r - \Delta n_2)} = -\,1.3744$$

$$R_3 = rR_1 = -\,5.1152$$



$R_1 = +\,2.5576$

$R_2 = -\,1.3744$

$R_3 = -\,5.1152$

$d_1 = \pm\,0$

$d_2 = \pm\,0$

Figure 138.

From the zero ray trace we find:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 8:

Compute the radii of a thin cemented achromatic triplet for which the following constants are given:

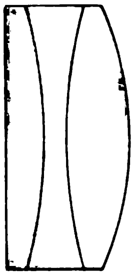
$f = 4.00$ inches $\quad R_1 = \pm\,\infty \quad R_2 = -\,R_3$ and $\Gamma = \pm\,0$

The glasses to be used have the following characteristics:

LENS	Δn	$_{C}n$	$_{D}n$	$_{F}n$	ν
1	0.00998	1.56956	1.5725	1.57954	57.4
2	0.01686	1.61217	1.6170	1.62903	36.6
3	0.00802	1.51462	1.5170	1.52264	64.5

$$R_4 = \frac{\{\Delta n_3\,(2n_2 - n_1 - n_3) - (n_3 - 1)\,(2\Delta n_2 - \Delta n_1 - \Delta n_3)\}\,f}{(2\Delta n_2 - \Delta n_1 - \Delta n_3) - (2n_2 - n_1 - n_3)\,f\Gamma} = -1.7731$$

$$R_2 = -R_3 = \frac{(2n_2 - n_1 - n_3)\,fR_4}{R_4 + (n_3 - 1)\,f} = -3.4753$$



$R_1 = \pm\infty$

$d_1 = \pm 0$

$R_2 = -3.4753$

$d_2 = \pm 0$

$R_3 = +3.4753$

$d_3 = \pm 0$

$R_4 = -1.7731$

Figure 139.

The focal lengths, computed with the zero rays, are as follows:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

Example 9:

Design a thin cemented achromatic triplet so that the system is symmetrical in shape.

We have:

$f = 4.00$ inches $\qquad R_1 = -R_4, \quad R_2 = -R_3 \qquad$ and $\qquad \Gamma = \pm 0$

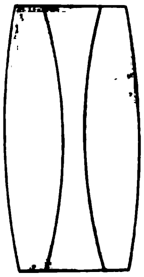
The glasses used in this combination have the following characteristics:

LENS	Δn	${}_C n$	${}_D n$	${}_F n$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61217	1.6170	1.62903	36.6
3	0.00802	1.51462	1.5170	1.52264	64.5

The following are the values of the radii if $n_1 = n_3$ and $\Delta n_1 = \Delta n_3$:

$$R_1 = -R_4 = \frac{2f\,\{(n_1 - 1)\,(\Delta n_2 - \Delta n_1) - (n_2 - n_1)\,\Delta n_1\}}{(\Delta n_2 - \Delta n_1) - (n_2 - n_1)\,f\Gamma} = +3.4102$$

$$R_2 = -R_3 = \frac{2(n_2 - n_1)\,fR_1}{R_1 - 2(n_1 - 1)\,f} = -3.7590$$



$R_1 = + 3.4102$

$d_1 = 0$

$R_2 = - 3.7590$

$d_2 = 0$

$R_3 = + 3.7590$

$d_3 = 0$

$R_4 = - 3.4102$

Figure 140.

Tracing the zero rays through the combination we find:

$\lambda =$	C	D	F
$f =$	4.0015	4.0000	4.0015

95. Single Thick Lens

It is possible to achromatize a single thick lens either with respect to the equivalent focal length or with respect with the back focal length. The computation of the radii for this type of lens is given in examples 10 and 11. It may be seen that these lenses require an extremely large thickness and, hence, they are of little practical importance.

96. Thick Cemented Doublet

In example 12 a thick cemented doublet has been computed with plane front and rear surfaces. In order that this lens be achromatic, the mean dispersion must be the same for both glasses, that is, $\Delta n_1 = \Delta n_2$. This principle is applied in the manufacture of ophthalmic bi-focal lenses but is of little importance in the design of optical systems for fire control instruments.

In examples 13 and 14 formulae are given for computing the radii for small collimators. Systems of this type are used in fire control instruments for a speedy location of the aiming point and, for this reason, a reference mark is etched on the plane surface of the collimator. In example 13 the collective lens is leading and in example 14 the dispersive lens is leading. The last surface is plane (as required) in both cases, and it is of interest to note that first radius and the total thickness are identical in both examples.

In examples 15 and 16 the computations for two cemented doublets are given. Both have a plane rear surface. In the first example, the doublet is achromatized with respect to the back focal lengths. In the second example, the equivalent focal lengths have been achromatized. Comparing the formulae given in these examples with those given in example 3 for a thin doublet, it will be noted that the equations for the thick doublet are more complicated than those for the thin doublet. The formulae becomes more complicated as soon as we impose different requirements on the construction of the doublet, or if we consider a cemented triplet. However, the system must not only have a specified focal length and be achromatic, but the lens must also be free from spherical aberration and coma as, for instance, in the case of telescope objective. In this case we design a thin system and, after introduction of thicknesses, all errors will be corrected simultaneously by means of the correction formulae. This method will be explained in the section on the design of telescope objectives which is included in volume II.

Example 10:

Determine the radii of a single thick lens so that ${}_Cf = {}_Ff$.

We have:

$${}_Df = 4.00 \text{ inches} \qquad r = -0.5 \qquad \text{and} \qquad \Gamma = {}_Cf - {}_Ff = \pm 0$$

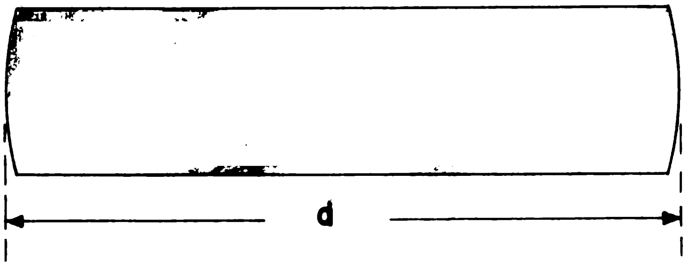
The glass will have the following characteristics:

Δn	${}_Cn$	${}_Dn$	${}_Fn$	ν
0.00802	1.51462	1.5170	1.52264	64.5

$$R_1 = \frac{({}_Dn - 1)\,(r - 1)\,({}_Dn - {}_Cn_Fn)\,{}_Df}{{}_Dn\,(1 - {}_Cn_Fn)\,r} = +2.4710$$

$$R_2 = rR_1 = -1.2355$$

$$d = \frac{{}_DnR_1\,\{R_2 - ({}_Dn - 1)\,(r - 1)\,{}_Df\}}{({}_Dn - 1)^2\,{}_Df} = 6.5440$$



$R_1 = +2.4710$

$R^2 = -1.2355$

$d = 6.5440$

Figure 141.

The equivalent focal lengths (f) and the back focal lengths (s′) for the various colors are given below.

$\lambda =$	C	D	F
$f =$	4.0001	4.0000	4.0001
$s' =$	0.4007	0.3897	0.3638

$\Gamma = \pm 0$

$\Gamma' = +0.0369$

Example 11:

Compute the radii of a single thick lens so that ${}_{C}s' = {}_{F}s'$.

We have:

$$f = 4.00 \text{ inches} \qquad \text{and} \qquad l' = {}_{C}s' - {}_{F}s' = \pm 0$$

$$\boxed{{}_{C}n = 1.51462 \qquad {}_{D}n = 1.5170 \qquad {}_{F}n = 1.52264}$$

It has been found by experience that, for a given f_D, the following equation will provide a ratio r between R_1 and R_2.

$$r = -\frac{({}_{C}n - {}_{F}n) \pm f_D\sqrt{{}_{C}n_{F}n}}{({}_{C}n - 1)\ ({}_{F}n - 1)}$$

The solution which follows, while it is a solution to the problem, is not unique and other solutions can be effected by different methods.

$$A = {}_{D}n^2\ ({}_{C}n - 1)\ ({}_{F}n - 1)\ r^2 = +\,315.7216$$

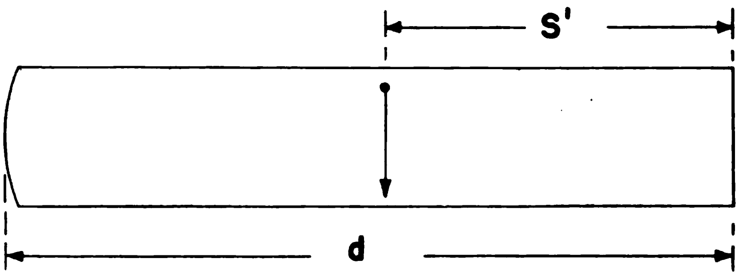
$$B = -\,{}_{D}n({}_{D}n - 1)_{D}fr\ \{({}_{D}n - 1)({}_{C}n_{F}n - 1) + ({}_{C}n - 1)({}_{D}n + 1)({}_{F}n - 1)(r - 1)\} = -\,1083.4267$$

$$C = ({}_{D}n - 1)^2(r - 1)_{D}f^2\ \{{}_{D}n({}_{C}n - 1)({}_{F}n - 1)(r - 1) + ({}_{D}n - 1)({}_{C}n_{F}n - {}_{D}n)\} = +\,929.4687$$

$$R_1 = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A} = +\,1.7158$$

$$R_2 = rR_1 = -\,38.7514$$

$$d = \frac{{}_{D}nR_1\ \{R_2 - ({}_{D}n - 1)\ (r - 1)\ {}_{D}f\}}{({}_{D}n - 1)^2{}_{D}f} = 21.4006$$



$R_1 = +\,1.7158$

$R_2 = -\,38.7514$

$d = 24.4006$

Figure 142.

The focal lengths and back focal lengths are as follows:

$\lambda =$	C	D	F
$f =$	4.0154	4.0000	3.9641
$s' =$	-15.3860	-15.3861	-15.3860

$\Gamma = +0.0513$

$\Gamma' = \pm 0$

NOTE: The images of a distant object are formed inside the lens at the distance s' from the rear surface.

Example 12:

Determine the second radius of a thick cemented doublet assuming that the front and rear surfaces are plane.

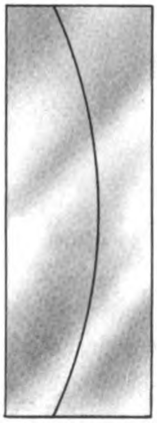
The following constants are known.

$${}_Df = 1 \text{ diopter} = 40 \text{ inches} \qquad R_1 = R_3 = \pm \infty$$

The condition imposed on the construction of the lens requires that $\Delta n_1 = \Delta n_2$, and for this reason the following glasses have been selected.

LENS	TYPE	Δn	${}_Cn$	${}_Dn$	${}_Fn$	ν
1	DC	0.01099	1.60866	1.6119	1.61965	55.7
2	BLF	0.01099	1.54508	1.5483	1.55607	49.9

$$R_2 = ({}_Dn_2 - {}_Dn_1)\ {}_Df = -2.544 \text{ inches}$$



$R_1 = \pm \infty$

$d_1 = 0.50$

$R_2 = -2.544$

$d_2 = 0.25$

$R_3 = \pm \infty$

The various focal lengths of this lens are tabulated below.

Figure 143.

$\lambda =$	C	D	F
$f =$	40.013	40.000	40.013
$s' =$	39.851	39.838	39.852

$\Gamma = \pm 0$

$\Gamma' = 0.001$

Example 13:

Determine the radii of a small achromatic collimator so that the cross lines may be etched on the plane rear surface.

The known constants are, ${}_Df = 1$ inch and $R_3 = \pm\infty$

The characteristics of the glasses to be used are as follows:

LENS	Δn	${}_Cn$	${}_Dn$	${}_Fn$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61218	1.6170	1.62904	36.6

$a_1 = {}_Cn_1 {}_Fn_1 = 2.306221$ $a_2 = {}_Cn_1 {}_Fn_2 = 2.467376$

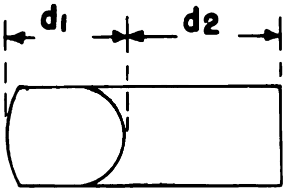
$$a_3 = {}_Fn_1 {}_Cn_2 = 2.454770$$

$$R_2 = \frac{\{a_1 \Delta n_2 ({}_Dn_2 - {}_Dn_1) - {}_Dn_1 ({}_Dn_2 - 1)(a_2 - a_3)\} {}_Df}{{}_Dn_1 \{a_2({}_Fn_1 - 1) - a_3({}_Cn_1 - 1)\}} = -0.19848$$

$$R_1 = \frac{{}_Dn_2 ({}_Dn_1 - 1) {}_Df\, R_2}{{}_Dn_1 R_2 - ({}_Dn_2 - {}_Dn_1) {}_Df} = 0.41369$$

$$d_1 = R_1 - R_2 = 0.61216$$

$$d_2 = \frac{{}_Cn_2 ({}_Cn_1 R_2 + d_1) R_2}{{}_Cn_1 ({}_Cn_2 - 1) R_2 + ({}_Cn_2 - {}_Cn_1) d_1} = 0.80194$$



$R_1 = +0.4137$

$R_2 = -0.1985$

$R_3 = \pm\infty$

$d_1 = 0.6122$

$D_2 = 0.8019$

Figure 144.

The equivalent and back focal lengths are:

$\lambda =$	C	D	F
$f =$	1.0004	1.0000	1.0004
$s' =$	0	-0.0002	0

$\Gamma = \Gamma' = \pm 0$

Example 14:

Design a collimator having the same constants as the one in the preceding example. The same glasses are to be used; however, the dispersive shall now be leading.

We have: ${}_Df = 1$ inch and $R_3 = \pm\infty$

The characteristics of the glasses are as follows:

LENS	Δn	${}_Cn$	${}_Dn$	${}_Fn$	ν
1	0.01686	1.61218	1.6170	1.62904	36.6
2	0.00802	1.51462	1.5170	1.52264	64.5

$$a_1 = {}_Cn_1\,{}_Fn_1 = 2.626306 \qquad a_2 = {}_Cn_1\,{}_Fn_2 = 2.454770$$

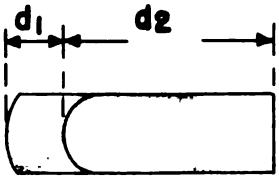
$$a_3 = {}_Fn_1\,{}_Cn_2 = 2.467376$$

$$R_2 = \frac{\{a_1\Delta n_2({}_Dn_2 - {}_Dn_1) - {}_Dn_1({}_Dn_2 - 1)(a_2 - a_3)\}\,{}_Df}{{}_Dn_1\,\{a_2({}_Fn_1 - 1) - a_3({}_Cn_1 - 1)\}} = 0.154874$$

$$R_1 = \frac{{}_Dn_2({}_Dn_1 - 1)\,{}_DfR_2}{{}_Dn_1R_2 - ({}_Dn_2 - {}_Dn_1)\,{}_Df} = 0.41367$$

$$d_1 = R_1 - R_2 = 0.258788$$

$$d_2 = \frac{{}_Cn_2({}_Cn_1R_2 + d_1)R_2}{{}_Cn_1({}_Cn_2 - 1)R_2 + ({}_Cn_2 - {}_Cn_1)d_1} = 1.155261$$



$R_1 = +\,0.4137$

$R_2 = +\,0.1549$

$R_3 = \pm\,\infty$

$d_1 = 0.2588$

$d_2 = 1.1553$

Figure 145.

The equivalent and back focal lengths for the various colors are:

$\lambda =$	C	D	F
$f =$	1.0004	1.0000	1.0004
$s' =$	± 0	— 0.0002	± 0

$\Gamma = \Gamma' = \pm 0$

Example 15:

Compute the radii of a thick cemented doublet with a plane rear surface so that ${}_Cs_2' = {}_Fs_2'$.

We have: ${}_Df = 2$ inches $\qquad \Gamma' = {}_Cs_2' - {}_Fs_2' = \pm 0 \qquad R_3 = \pm\,\infty$

$d_1 = 0.360 \qquad d_2 = 0.075$

The characteristics of the glasses are as follows:

LENS	Δn	$_Cn$	$_Dn$	$_Fn$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61218	1.6170	1.62904	36.6

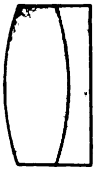
$$A = {}_Dn_2^2({}_Dn_1 - 1)^2({}_Dn_1\Delta n_2 - {}_Dn_2\Delta n_1)d_1 = 0.0031722$$

$$B = {}_Dn_{1D}n_2^2 \left\{ {}_Dn_1({}_Dn_2 - 1)\Delta n_1 - {}_Dn_1({}_Dn_1 - 1)\Delta n_2 - ({}_Dn_1 - 1)(\Delta n_2 - \Delta n_1)d_{1D}\phi \right\}$$
$$= -0.0259376$$

$$C = {}_Dn_1^2{}_Dn_2^2(\Delta n_2 - \Delta n_1){}_D\phi - {}_Dn_1^2({}_Dn_2 - {}_Dn_1)\left\{ \Delta n_2 d_2 + {}_Dn_2^2\,\Gamma' \right\}{}_D\phi^2 = 0.0265230$$

$$R_1 = -\frac{2A}{B \pm \sqrt{B^2 - 4AC}} = +0.8348773$$

$$R_2 = \frac{({}_Dn_2 - {}_Dn_1){}_Df\left\{ {}_Dn_1R_1 - ({}_Dn_1 - 1)d_1 \right\}}{{}_Dn_1\left\{ R_1 - ({}_Dn_1 - 1){}_Df \right\}} = -0.715325$$



$R_1 = +0.8349$

$R_2 = -0.7153$

$R_3 = \pm\infty$

$d_1 = 0.360$

$d_2 = 0.075$

Figure 146.

$\lambda =$	C	D	F
$f =$	2.0013	2.0000	2.0020
$s_2' =$	1.6604	1.6597	1.6604

$\Gamma = -0.0007$

$\Gamma' = \pm 0$

Example 16:

Compute the radii of a thick cemented doublet with a plane rear surface so that $_C\phi = {}_F\phi$.

We have: $_Df = 2$ **inches** $\Gamma = {}_F\phi - {}_C\phi = \pm 0$ $R_3 = \pm\infty$

The characteristics of the glasses are:

LENS	Δn	$_Cn$	$_Dn$	$_Fn$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01686	1.61218	1.6170	1.62904	36.6

$$a = {}_{C}n_{1F}n_1(\Delta n_2 - \Delta n_1) = 0.020387 \qquad b = {}_{C}n_{1F}n_1({}_{D}n_2 - {}_{D}n_1) = 0.230622$$

$$c = {}_{D}n_1({}_{C}n_{1F}n_2 - {}_{F}n_{1C}n_2) = 0.019125$$

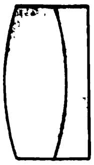
$$A = ({}_{D}n_1 - 1)d_1 \{ {}_{D}n_1 a - \Delta n_1 b - c\} = 0.001852$$

$$B = -{}_{C}n_{1D}n_{1F}n_1 \{ ({}_{D}n_1 - 1)\Delta n_2 - ({}_{D}n_2 - 1)\Delta n_1\} - ({}_{D}n_1 a - c)d_{1D}\phi + ({}_{D}n_1 - 1)bd_1\Gamma$$
$$= -- 0.0153080$$

$$C = {}_{D}n_1(a_D\phi - b\Gamma) = 0.015464$$

$$R_1 = -\frac{2A}{B \pm \sqrt{B^2 - 4AC}} = +0.8483502$$

$$R_2 = \frac{({}_{D}n_2 - {}_{D}n_1){}_{D}f\{ {}_{D}n_1 R_1 - ({}_{D}n_1 - 1)d_1\}}{{}_{D}n_1\{R_1 - ({}_{D}n_1 - 1){}_{D}f\}} = -0.7817608$$



$R_1 = +0.8484$

$R_2 = -0.7818$

$R_3 = \pm \infty$

$d_1 = 0.360$

$d_2 = 0.075$

Figure 147.

$\lambda =$	C	D	F
$f =$	2.0008	2.0000	2.0008
$s_2' =$	1.6658	1.6644	1.6634

$\Gamma = \pm 0$

$\Gamma' = +0.0024$

To conclude this section a method will be given whereby the chromatic aberration in a doublet or triplet can be eliminated by a small change in the rear surface of the system. This method is fully explained in example 17.

Example 17:

Compute the magnitudes of the residual chromatic aberrations of a cemented doublet of which the following constants are known.

$R_1 = +2.44905$

$R_2 = -1.64339$

$R_3 = -5.35213$

$d_1 = 0.190$

$d_2 = 0.085$

The characteristics of the glasses are as follows:

LENS	Δn	$_C n$	$_D n$	$_F n$	ν
1	0.00802	1.51462	1.5170	1.52264	64.5
2	0.01926	1.64354	1.6490	1.66280	33.7

Trace zero rays through the combination for light corresponding to the C and F lines of the spectrum and determine the magnitudes of the residual chromatic aberrations. Below are given the computations for the last surface of the doublet.

$\lambda =$	C	F
$a_3 =$	0.138356	0.134880
$b_3 =$	0.120239	0.123838
$_3' =$	0.258595	0.258712
$s_3' =$	3.867050	3.865212
$h_3 =$	0.966723	0.966705
$\alpha_3 =$	0.249990	0.250104
$f =$	4.000160	3.998336

The longitudinal chromatic aberration is:

$$CA = {}_C s_3' - {}_F s_3' = +\,0.001838$$

The color magnification is:

$$CM = {}_C f - {}_F f = +\,0.001824$$

Therefore, the doublet is chromatically under-corrected. In order to correct these errors we change the last radius (R_3) by applying the following equation:

$$R_3 = \frac{\Delta n_2}{{}_F a_3 - {}_C a_3} = -\frac{0.01926}{0.003476} = -\,5.540851 \tag{559}$$

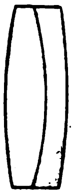
Repeat the zero ray trace but substitute the new value for R_3.

$\lambda =$	C	F
$a_3 =$	0.138356	0.134880
$b_3 =$	0.116144	0.119620
$_3' =$	0.254500	0.254500
$s_3' =$	3.929273	2.929273
$h_3 =$	0.966723	0.966705
$\alpha_3 =$	0.246031	0.246026
$f =$	4.064528	4.064611

$$CA = \pm\,0$$

$$CM = -\,0.000083$$

The final constants of the chromatically corrected doublet are as follows:



$R_1 = +2.44905$

$R_2 = -1.64339$

$R_3 = -5.54085$

$d_1 = 0.190$

$d_2 = 0.085$

Figure 148.

In the case of a cemented triplet, equation (559) is written as follows:

$$R_4 = \frac{\Delta n_3}{{}_Fa_4 - {}_Ca_4} \tag{560}$$

SECTION VIII

ABERRATIONS OF THE IMAGE

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97. Introduction

In section V we have dealt with the laws and formulae relating to the position and size of the image of a given object formed by a system of centered spherical surfaces; however, the quality of the image was not mentioned. All images suffer from certain aberrations. One of these, the chromatic aberration, has been discussed in section VII and will not be considered here.

Previously we considered only rays within the paraxial region, such as paraxial and zero rays. However, rays of light diverging from an object point at a fairly large angle will not converge to a single point in the image after refraction by a single lens, but will spread out over a small area. The image of an extended object may be thought of as being a mosaic made up of small blocks, each block representing the image of a point in the object; hence, the image lacks sharpness so that fine details in the object cannot be distinguished. This lack of sharpness in the image is due to the presence of residual aberrations, that is, the aberrations of sphericity. According to L. von Seidel there are five such errors known as *"SPHERICAL ABERRATION," "COMA," "ASTIGMATISM," "FIELD CURVATURE,"* and *"DISTORTION."*

There is no optical system in existence in which all the errors are completely eliminated, and it is the problem of the optical engineer to design an optical system in which the residual aberrations are well balanced and within their tolerances. The five *"Seidel sums"* are expressed by S_I, S_{II}, S_{III}, S_{IV}, and S_V, and the conditions that each of the five errors will disappear is that the sums be made equal to zero for a definite height of incidence. If $S_I = 0$, the axial spherical aberration is eliminated; if $S_{II} = 0$, the coma is corrected. Astigmatism, field curvature, and distortion are each separately corrected when S_{III}, S_{IV}, and S_V, respectively, are equal to zero. We then obtain an image of a plane object which is well defined, flat, and undistorted, but only for one position of the object. Numerous formulae have been derived from the five Seidel sums which are applied in computing thin lens systems in which the errors are eliminated for any object distance and any height of incidence.

The computing of the magnitude of the residual errors by the trigonometric ray tracing formulae has been shown in section VI. The formulae given there show that the diaphragm distance x_1 influences the magnitudes of coma, astigmatism, field curvature, and distortion. For this reason we will discuss the position of the diaphragm first.

98. Diaphragms, Entrance and Exit Pupils

The bundle of rays utilized in the formation of the image is limited by diaphragms. One such diaphragm, or aperture stop, is the lens cell; in addition to cell stops every optical system has one or more diaphragms (perforated screens) whose function is to limit the size of the bundle of rays, thus providing a means for reducing the residual aberrations and preventing stray light from reaching the image. In instruments for fire control, the apertures of the diaphragms must be sufficiently large so that the illumination of the image or the required field of view is not reduced. The stop which is most effective in limiting the size of a bundle of rays is known as the *"APERTURE STOP."* Stops are classified as *"FRONT," "REAR,"* or *"INTERIOR"* diaphragms, depending on whether they lie in front, in back, or inside the optical system.

In figure 149 is shown the diaphragm at a distance x in front of the lens. This stop will limit the diameter of the bundle of rays emanating from the object, T, which can pass through the lens; it is the *"ENTRANCE PUPIL"* of the lens and the diameter of its aperture is denoted by EP. The image of this stop is located at a distance x′ in front of the lens. The aperture of the stop image is termed the *"EXIT PUPIL"* of the lens and is designated by AP. The planes containing the entrance and exit pupils are always conjugate. Since the peripheries of the EP and the AP are always conjugate, it follows that every incident ray which grazes the edge of the EP will, after refraction by the lens, and if produced, graze the edge of the AP. If the lens cell is the diaphragm, then the EP and AP are coincident and lie within the lens.

The lens shown in figure 150 is provided with a rear stop at a distance x behind the lens. The image of this stop is formed at the distance x′ from the lens. Rays coming from any point of the object, T, which go through the stop must have originally been aimed at the image of the stop. Therefore, in this case the physical diaphragm (the distance x behind the lens) serves as the exit pupil while the entrance pupil is located in the image plane of the diaphragm. The relations between the various constants are mathematically expressed as follows:

$$x' = \frac{fx}{f + x} \tag{561}$$

$$AP = \frac{x'}{x}\,(EP) \tag{562}$$

In figure 151 are shown two thin lenses of equal focal lengths, separated by the distance D. In the center of the system is placed a physical diaphragm whose clear aperture is 2ρ. The image of this interior stop, formed by the front lens, will be the entrance pupil of the system when viewed from the object point, T. The image of the stop, produced by the rear lens, will be the exit pupil of the system when viewed from the image point, T′.

Example 1:

a. The lens shown in figure 149 has a focal length of 1.5 inches. The diaphragm, whose clear aperture is 0.50 inch, is located at a distance of 0.50 inch in front of the lens. What is the image distance x′ and what is the magnitude of the AP?

We have, $f = 1.50$, $x = -0.50$, and $EP = 0.50$

With the aid of equations (561) and (562) we find,

$$x' = \frac{(1.5)(-0.5)}{1.5 - 0.5} = -0.75 \qquad AP = \frac{0.75}{0.50}\,(0.50) = 0.75$$

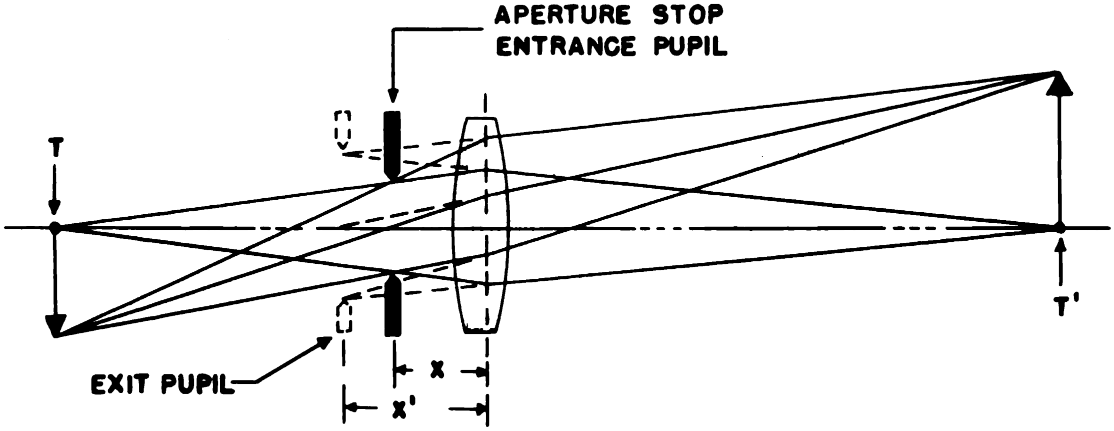


Figure 149. Lens with front stop.

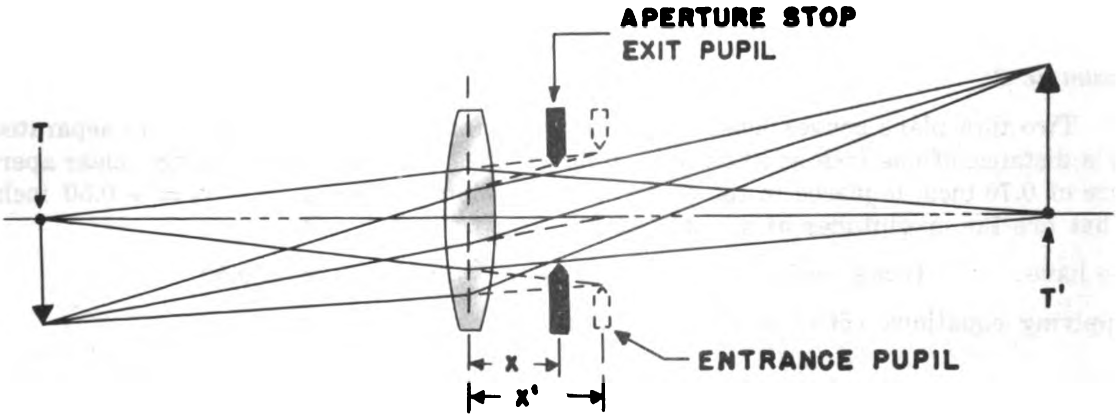


Figure 150. Lens with rear stop.

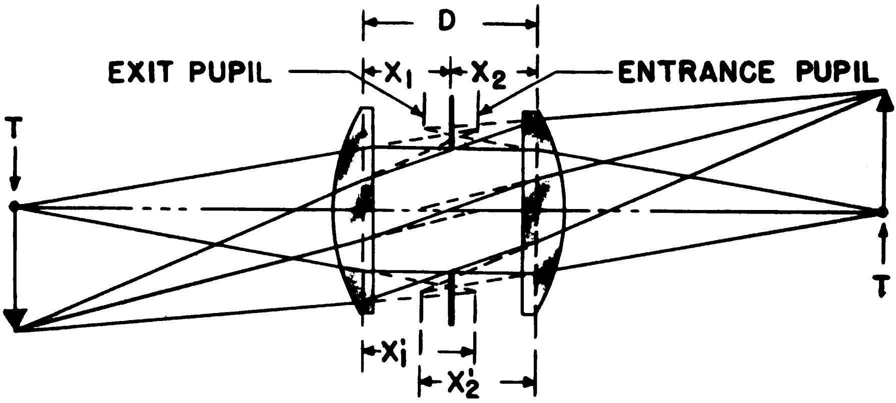


Figure 151. Lens with interior stop.

b. The same as above except that the physical diaphragm is located at a distance of 0.50 inch behind the lens as shown in figure 150.

We have, $f = 1.50$, $x = +0.50$, and $AP = 0.50$

In this case we apply the following formulae:

$$x' = \frac{fx}{f - x} = +0.75 \tag{563}$$

$$EP = \frac{x'}{x}(AP) = 0.75 \tag{564}$$

Example 2:

Two thin plane convex lenses, each having a focal length of 2 inches, are separated by a distance of one inch as illustrated in figure 151. A diaphragm, having a clear aperture of 0.70 inch, is placed in the center of the system so that $x_1 = -x_2 = +0.50$ inch. What are the magnitudes of x_1' and AP?

We have: $f_1 = f_2 = 2.00$ $2\rho = 0.70$ and $x_1 = 0.50$

Applying equations (563) and (564) we find:

$$x_1' = -x_2' = \frac{(2)\,(0.50)}{2 - 0.50} = +0.667$$

$$AP = EP = \frac{(2\rho)x'}{x} = 0.934 \tag{565}$$

The rays which pass through the center of the aperture stop are called the *"PRINCIPAL RAYS,"* or the *"CHIEF RAYS."*

In figure 152 are shown two thin lenses mounted in the tube a b c d. The lenses are separated by the distance D. F_1 and F_2 are the focal points of the two lenses. There are no diaphragms in the ordinary sense; the edges of the lenses act as such.

Two cases must be considered:

Case I:

The rim of the second lens is the exit pupil of the system. This case is illustrated in figure 152. We have $AP = 2h_2$ and

$$s = \frac{Df_1}{D - f_1} \tag{566}$$

$$EP = \frac{2h_1(s - f_1)}{f_1 + s} \tag{567}$$

$$L = \frac{(h_1 - EP/2)s}{h_1} = \frac{(h_1 + EP/2)f_1}{h_1} \tag{568}$$

Case II:

The rim of the first lens is the entrance pupil of the system hence, $EP = 2h_1$ and

$$s' = \frac{Df_2}{D - f_2} \tag{569}$$

$$AP = a'c' = \frac{2h_2(s' - f_2)}{f_2 + s'} \tag{570}$$

$$L' = \frac{(h_2 - AP/2)s'}{h_2} = \frac{(h_2 + AP/2)f_2}{h_2} \tag{571}$$

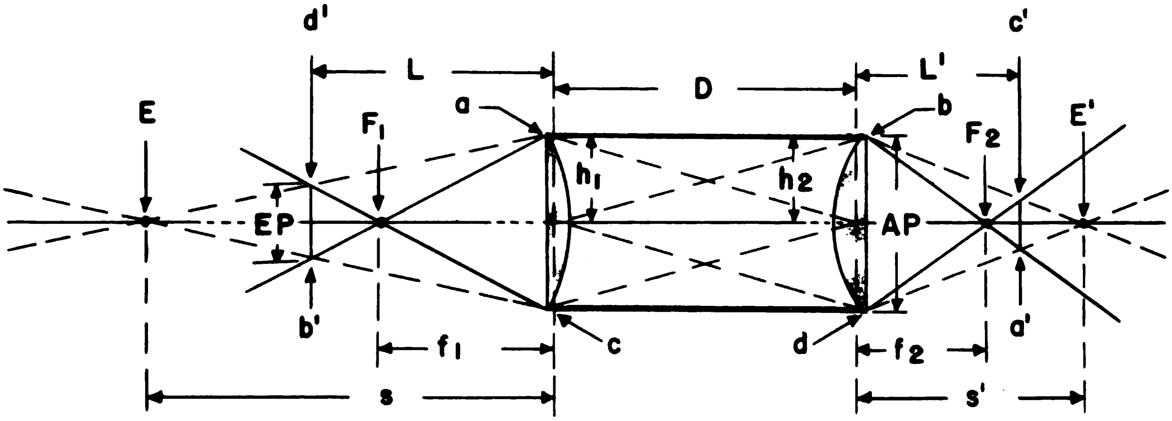


Figure 152. Location of EP when AP is cell of second lens.

Example 3:

The thin lenses shown in figure 152 have focal lengths of 1 inch and 0.75 inch, respectively; their clear apertures are equal to 1 inch and they are separated by a distance of 1.75 inches.

We have $f_1 = 1.00$, $f_2 = 0.75$, $D = 1.75$, and $h_1 = h_2 = 0.50$

Case I:

The cell of the second lens is the exit pupil of the system, hence, $AP = 2h_2 = 1.00$. b′d′ is the image of the rear lens produced by the front lens and, therefore, b′d′ is the entrance pupil of the system. Applying equations (566), (567), and (568), we find:

$$s = 2.333 \qquad EP = b'd' = 0.40 \qquad L = 1.40$$

Case II:

The cell of the first lens is the entrance pupil of the system, hence, $EP = 2h_1 = 1.00$. a′c′ is the image of the front lens produced by the rear lens and, hence, a′c′ is the exit pupil in this case. With the aid of equations (569), (570), and (571), we find,

$$s' = 1.313 \qquad AP = a'c' = 0.273 \qquad L' = 0.955$$

NOTE: The image b′d′ of the rear aperture bd of the tube formed by the front lens is the entrance pupil of the system; the front aperture of the tube is the entrance pupil only when the object point lies between the points E and F_1.

99. Entrance and Exit Windows

The entrance window is that diaphragm in object space which subtends the smallest angle at the entrance pupil. In figure 149 the cell of the lens limits the aperture of the bundle of incident principal rays just as the entrance pupil limits the bundle of incident rays proceeding from any object point and, hence, the cell of the lens is the entrance window. In the upper diagram of figure 153 is shown the diaphragm which is the entrance pupil for the lens. Bundles of rays coming from the object points A, B, and T will pass through this stop. A second stop is placed between the entrance pupil and the lens which will stop the bundle of rays proceeding from the object point B, hence, this stop is the entrance window. In the lower diagram are shown the positions of the entrance window and its conjugate image, which is the exit window of the lens. A number of principal rays are shown passing through the lens.

In figure 154 is shown a telescope objective in a cell. The cell is the entrance pupil and the entrance window of the telescope. Now we place a cylindrical sun-shade over the objective. If the inside diameter 2r of the sun-shade is 0.75 inch, the clear aperture $2r_1$ is 0.65 inch, and the true field of view 2α of the telescope is 6°, then the maximum length L of the sun-shade possible without vignetting is computed as follows,

$$L = (r - r_1) \text{ cotan } \alpha = 1.9081 \text{ inches} \tag{572}$$

If a longer sun-shade is required, or if the true field is increased, we must increase the inside diameter of the sun-shade or substitute a funnel shaped sun-shade.

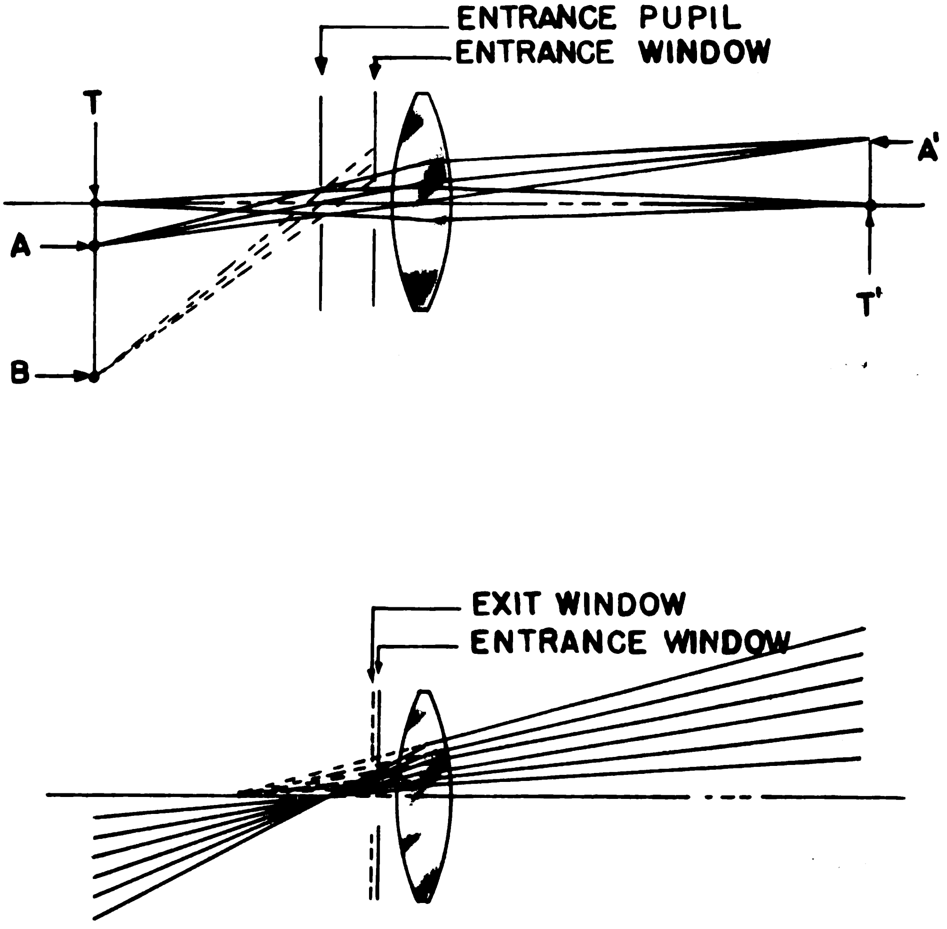


Figure 153. Entrance and exit windows.

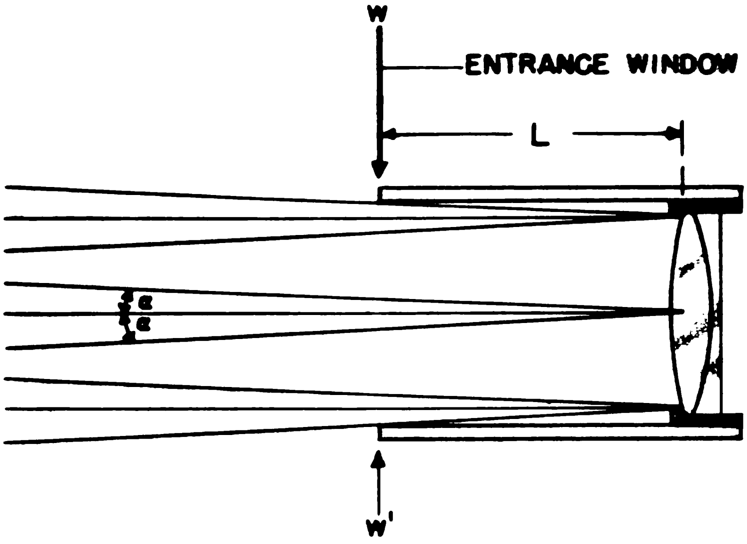


Figure 154. Entrance window when sun-shade is used.

100. Seidel Sums

a. Abbreviations.

The following abbreviations are used frequently in this section:

f = focal length of the lens and $\phi = \frac{1}{f}$
h_1 = distance from axis to point of incidence at first surface
R_v = radius of curvature of the vth surface and $\rho_v = \frac{1}{R_v}$
s_1 = the object distance and $\sigma_1 = \frac{1}{s_1}$
s_2' = the back focal length of the lens
x_1 = the diaphragm distance and $\xi_1 = \frac{1}{x_1}$

The contributions to the various errors by the vth surface are designated as follows:

A_v = contribution to spherical aberration
B_v = contribution to coma
C_v = contribution to astigmatism
D_v = contribution to distortion
P_v = contribution to field curvature

For a single thin lens the Seidel sums are abbreviated as follows:

S_I = spherical aberration
S_{II} = coma

S_{III} = curvature of the primary image
S_{IV} = curvature of the secondary image
S_V = distortion
S_{IIIa} = astigmatism
S_{IVa} = mean field curvature

The angular magnitudes of the residual aberrations are designated as follows:

AC = angular coma
AD = angular distortion
APC = angular primary curvature
ASA = angular spherical aberration
ASC = angular secondary curvature

The linear magnitudes of the residual aberrations are abbreviated as follows:

AST = the astigmatism
DIST = the distortion
FC = the mean field curvature
SA = the longitudinal spherical aberration
R_s = radius of the secondary image
R_t = radius of the primary image
ΔS = distance from image plane to secondary focal point
Δt = distance from image plane to primary focal point

b. Derivation of formulae.

Constants required in computing the Seidel sums for a single thin lens:

$$Q_v = n_{v-1}(\rho_v - \sigma_v) \tag{573}$$

$$\Delta\left(\frac{\sigma}{n}\right)_v = \frac{\sigma_v'}{n_v} - \frac{\sigma_v}{n_{v-1}} \tag{574}$$

$$\tau_v = \frac{1}{n_{v-1}(\rho_v - \sigma_v)} \tag{575}$$

$$\Delta\left(\frac{1}{n}\right)_v = \frac{1}{n_v} - \frac{1}{n_{v-1}} \tag{576}$$

$$A_v = Q_v^2\,\Delta\left(\frac{\sigma}{n}\right)_v \tag{577}$$

$$B_v = \tau_v A_v \tag{578}$$

$$C_v = \tau_v B_v \tag{579}$$

$$D_v = \tau_v (C_v + P_v) \tag{580}$$

$$P_v = -\,\Delta\left(\frac{1}{n}\right)_v \rho_v \tag{581}$$

Eliminate ρ_2 by allowing ρ_2 to equal $\rho_1 - \frac{\phi}{n-1}$ in the above equation. (582)

Then:

$$\sigma_1' = \sigma_2 = \frac{n-1}{n}\rho_1 + \frac{1}{n}\sigma_1 \tag{583}$$

$$\sigma_2' = \phi + \sigma_1 \tag{584}$$

$$\Delta\left(\frac{\sigma}{n}\right)_1 = \frac{n-1}{n^2}\rho_1 - \frac{n^2-1}{n^2}\sigma_1 \tag{585}$$

$$\Delta\left(\frac{\sigma}{n}\right)_2 = -\frac{n-1}{n^2}\rho_1 + \frac{n^2-1}{n^2}\sigma_1 + \phi \tag{586}$$

$$Q_1 = \rho_1 - \sigma_1 \tag{587}$$

$$Q_2 = \rho_1 - \frac{n}{n-1}\phi - \sigma_1 \tag{588}$$

$$A_1 = +\frac{n-1}{n^2}\rho_1^3 - \left\{\frac{2(n-1)}{n^2} + \frac{n^2-1}{n^2}\right\}\rho_1^2\sigma_1 + \left\{\frac{n-1}{n^2} + \frac{2(n^2-1)}{n^2}\right\}\rho_1\sigma_1^2 - \frac{n^2-1}{n^2}\sigma_1^3$$

$$A_2 = -\frac{n-1}{n^2}\rho_1^3 + \left\{\frac{2(n-1)}{n^2} + \frac{n^2-1}{n^2}\right\}\rho_1^2\sigma_1 - \left\{\frac{n-1}{n^2} + \frac{2(n^2-1)}{n^2}\right\}\rho_1\sigma_1^2 + \frac{n^2-1}{n^2}\sigma_1^3 + \frac{n+2}{n}\phi\rho_1^2 - \frac{2n+1}{n-1}\phi^2\rho_1 - \frac{4(n+1)}{n}\phi\rho_1\sigma_1 + \left(\frac{n}{n-1}\right)^2\phi^3 + \frac{3n+1}{n-1}\phi^2\sigma_1 + \frac{3n+2}{n}\phi\sigma_1^2$$

$$\left.\begin{aligned} A = A_1 + A_2 = {} & \frac{n+2}{n}\phi\rho_1^2 - \frac{2n+1}{n-1}\phi^2\rho_1 - \frac{4(n+1)}{n}\phi\rho_1\sigma_1 + \\ & \left(\frac{n}{n-1}\right)^2\phi^3 + \frac{3n+1}{n-1}\phi^2\sigma_1 + \frac{3n+2}{n}\phi\sigma_1^2 \end{aligned}\right] \tag{589}$$

Let:

$$\left.\begin{aligned} & a = \frac{n+2}{n}\phi \qquad b = -\left[\frac{2n+1}{n-1}\phi + \frac{4(n+1)}{n}\sigma_1\right]\phi \\ & c = \left(\frac{n}{n-1}\right)^2\phi^3 + \frac{3n+1}{n-1}\phi^2\sigma_1 + \frac{3n+2}{n}\phi\sigma_1^2 \end{aligned}\right] \tag{590}$$

then, in the absence of spherical aberration,

$$A = a\rho_1^2 + b\rho_1 + c = 0 \tag{591}$$

hence:

$$\rho_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{592}$$

In order to compute a single thin lens with the minimum spherical aberration we differentiate equation (589) with respect to ρ_1 and we find,

$$\frac{\delta A}{\delta \rho_1} = \frac{2(n+2)}{n}\rho_1 - \frac{2n+1}{n-1}\phi - \frac{4(n+1)}{n} = 0 \tag{593}$$

$$\text{hence,}\ \rho_1 = \frac{n(2n+1)}{2(n-1)(n+2)}\phi + \frac{2(n+1)}{n+2}\sigma_1 \tag{594}$$

$$\text{when}\ \sigma_1 = \pm 0,\ \text{then}\ \rho_1 = \frac{n(2n+1)}{2(n-1)(n+2)}\phi \tag{595}$$

$$\text{when}\ \sigma_1 = -\frac{\phi}{2},\ \text{then}\ \rho_1 = -\rho_2 = \frac{1}{2(n-1)}\phi \tag{596}$$

$$\text{when}\ \sigma_1 = -\phi,\ \text{then}\ \rho_1 = \left[\frac{n(2n+1) - 4(n^2-1)}{2(n-1)(n+2)}\right]\phi \tag{597}$$

The minimum value of A that a lens of this type will have is computed as follows:

$$A_{(Min)} = \frac{n(4n-1)}{4(n-1)^2(n+2)}\phi^3 \tag{598}$$

If the single thin lens, having the minimum spherical aberration, is to be a plane-convex lens with its convex surface toward the distant object, the refractive index of the glass must be equal to

$$n = \frac{1+\sqrt{33}}{4} \tag{599}$$

The object distance s_1 for which a single thin lens is free from spherical aberration is computed as follows:

$$s_1 = \frac{2(n-1)f}{4n-1}\left\{(n-1) \pm \sqrt{n(n+2)}\right\} \tag{600}$$

The angular spherical aberration is computed thus:

$$\text{ASA (in radians)} = \frac{h_1^3}{2}A \quad \text{or,} \quad \text{ASA (in minutes)} = 1718.87\ h_1^3 A \tag{601}$$

The linear spherical aberration is:

$$SA = \frac{h_1^2 s_2'^2}{2} A \tag{602}$$

A single thin lens with a minimum amount of spherical aberration is also known as *"THE LENS OF THE BEST FORM."*

If

$$B = \tau_1 A_1 + \tau_2 A_2 \tag{603}$$

$$\text{or}\quad B = Q_1 \Delta \left(\frac{\sigma}{n}\right)_1 + Q_2 \Delta \left(\frac{\sigma}{n}\right)_2 \tag{604}$$

then substituting into equations (585), (586), (587), and (588), we find:

$$B = B_1 + B_2 = \frac{n+1}{n} \phi\rho_1 - \frac{n}{n-1} \phi^2 - \frac{2n+1}{n} \phi\sigma_1 \tag{605}$$

If we put $B = \pm 0$, then

$$\rho_1 = \frac{n^2}{n^2-1} \phi + \frac{2n+1}{n+1} \sigma_1 \tag{606}$$

When the object is at infinity σ_1 equals ± 0 and the last equation reduces to:

$$\rho_1 = \frac{n^2}{n^2-1} \phi \tag{607}$$

If the coma is to be corrected by the use of a diaphragm when $\sigma_1 = \pm 0$, then the diaphragm distance is computed as follows:

$$x_1 = -\frac{(n-1) f \{(n-1) \pm 2n\sqrt{n+2}\}}{n(4n-1)} \tag{608}$$

If we put

$$g_1 = \mp \frac{(n^2-1) \{1 + 2x_1(2\sigma_1 + \phi)\}}{(n+2)x_1} \tag{609}$$

$$\rho_1 = \frac{g_1 + \phi}{2(n-1)} \quad \text{and} \quad \rho_2 = \frac{g_1 - \phi}{2(n-1)} \tag{610}$$

and substitute the values of ρ into equations (591) and (605), we find:

$$\text{Coma} = A + (\sigma_1 - \xi_1) B = 0 \tag{611}$$

If the single thin lens is to be spherically corrected and free from coma then:

$$x_1 = -\frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{612}$$

where

$$\alpha = -n^2(4n-1)\phi^2 s_1^2 + 2n(n-1)^2(2n+1)\phi s_1 + (n-1)^2\{n(5n+6)+1\}$$

$$\beta = -2(n-1)^2\{n(n+4)+1+n\phi s_1\}s_1$$

$$\gamma = +(n-1)^2\{n(n+2)+1\}s_1^2$$

If we let the discriminant in equation (612) be equal to zero, that is, $\beta^2 - 4\alpha\gamma$ equals 0, we find the object distance for which the spherical aberration is eliminated as follows:

$$s_1 = \frac{(n-1)^2 \pm (n^2+1)}{2n\phi}, \text{ that is, } s_1 = +\frac{n-1}{\phi} \text{ or } s_1 = -\frac{n-1}{n\phi} \tag{613}$$

We put

$$g_2 = \pm\frac{\{n+2(n+1)^2\}\phi}{n+2} \tag{614}$$

$$\rho_1 = \frac{g_2+\phi}{2(n-1)} \text{ and } \rho_2 = \frac{g_2-\phi}{2(n-1)} \tag{615}$$

The angular coma is computed thus:

$$AC = -\frac{3h_1^2 \tan\alpha_0}{2} B \tag{616}$$

$$\text{and coma} = -\frac{3h_1^2 z s_2' \tan\alpha_0}{2} B \tag{617}$$

When the object is at infinity and the single thin lens is to be free from astigmatism we have $S_{VIa} = 0$. In this case the diaphragm distance is derived from the equation:

$$x_1^2 A - 2x_1 B + C = 0 \tag{618}$$

(for a thin lens $C = \phi$) hence,

$$x_1 = \frac{B \pm \sqrt{B^2 - A\phi}}{A} \tag{619}$$

If the single thin lens is to have a minimum field curvature we put:

$$g_3 = -\frac{(n+1)(2n-1)}{n-1}\phi \tag{620}$$

$$x_1 = -\frac{2(n-1)^2}{n(4n-1)\phi} \tag{621}$$

$$\rho_1 = \frac{g_3+\phi}{2(n-1)} \text{ and } \rho_2 = \frac{g_3-\phi}{2(n-1)} \tag{622}$$

$$\text{and} \quad S_{III} = \frac{2n-1}{4n(4n-1)}\,\phi \tag{623}$$

For convenience, the formulae required to compute the residual aberration in a single thin lens are given here.

$$\left.\begin{aligned} A = \frac{n+2}{n}\,\phi\rho_1^2 - \frac{2n+1}{n-1}\,\phi^2\rho_1 - \frac{4(n+1)}{n}\,\phi\rho_1\sigma_1 + \left(\frac{n}{n-1}\right)^2\phi^3 + \\ \frac{3n+1}{n-1}\,\phi^2\sigma_1 + \frac{3n+2}{n}\,\phi\sigma_1^2 \end{aligned}\right] \tag{589}$$

$$B = \frac{n+1}{n}\,\phi\rho_1 - \frac{n}{n-1}\,\phi^2 - \frac{2n+1}{n}\,\phi\sigma_1 \tag{605}$$

$$C = \phi \tag{579}$$

$$D = \pm\, 0 \tag{580}$$

$$P = \frac{\phi}{n} \tag{581}$$

The Seidel Sums

$$\left.\begin{aligned} S_I &= A \\ S_{II} &= \frac{A}{\sigma_1 - \xi_1} + B \\ S_{III} &= \frac{3A}{(\sigma_1 - \xi_1)^2} + \frac{6B}{\sigma_1 - \xi_1} + (3C + P) \\ S_{IIIa} &= \frac{A}{(\sigma_1 - \xi_1)^2} + \frac{2B}{\sigma_1 - \xi_1} + C \\ S_{IV} &= \frac{A}{(\sigma_1 - \xi_1)^2} + \frac{2B}{\sigma_1 - \xi_1} + (C + P) \\ S_{IVa} &= \frac{2A}{(\sigma_1 - \xi_1)^2} + \frac{4B}{\sigma_1 - \xi_1} + (2C + P) \\ S_V &= \frac{A}{(\sigma_1 - \xi_1)^3} + \frac{3B}{(\sigma_1 - \xi_1)^2} + \frac{(3C + P)}{\sigma_1 - \xi_1} + D \end{aligned}\right] \tag{624}$$

The Angular Errors

$$\left.\begin{aligned}
\mathrm{ASA} &= \frac{h_1^3}{2} S_I \\
\mathrm{AC} &= \frac{3h_1^2 \tan \alpha_0}{2} S_{II} \\
\mathrm{APC} &= - h_1 \tan^2 \alpha_0 \frac{(S_{III} + S_{IV})}{2} \\
\mathrm{ASP} &= - h_1 \tan^2 \alpha_0 S_{IV} \\
\mathrm{AD} &= \tan^3 \alpha_0 S_V \\
\mathrm{AST} &= - \frac{s_2'^2 \tan^2 \alpha_0}{2} S_{IIIa}
\end{aligned}\right] \qquad (625)$$

The Linear Errors

$$\left.\begin{aligned}
\mathrm{SA} &= \frac{h_1^2 s_2'^2}{2} (S_I) \\
\mathrm{Coma} &= - \frac{3h_1^2 s_2' \tan \alpha_0}{2} (S_{II}) \\
\Delta t &= - \frac{s_2'^2 \tan^2 \alpha_0}{2} (S_{III})_a \\
\Delta S &= - \frac{s_2'^2 \tan^2 \alpha_0}{2} (S_{IV}) \\
\mathrm{DIST} &= s_2' \tan^3 \alpha_0 (S_V) \\
\mathrm{AST} &= - s_2'^2 \tan^2 \alpha_0 (S_{IVa} - S_{IV}) \\
\mathrm{FC} &= - \frac{s_2'^2 \tan^2 \alpha_0}{2} (S_{IVa}) \\
R_t &= - \frac{1}{S_{III}} \\
R_s &= - \frac{1}{S_{IV}}
\end{aligned}\right] \qquad (626)$$

NOTE: When $s_1 = \infty$, substitute f for s_2' in equation (626).

Example 4:

Compute a lens of the best form. The focal length of the lens is to be 4 inches and the refractive index of the glass 1.5170. Determine the radii of the lens when the object distances are:

(a) $s_1 = \pm \infty$, (b) $s_1 = -2f$, and (c) $s_1 = -f$

We have: $f = 4.000$ $\phi = 0.25$ $n = 1.5170$ $h_1 = 0.25$ $\alpha_o = 1°30'$

Case I: $s_1 = \infty$ hence, $\sigma_1 = 0$

Substitute these values into equations (595), (582), (584), (598), (601), and (602), and find:

$\rho_1 = +0.420696$ $R_1 = +2.377$ $\sigma_2' = +0.25$

$A_{Min} = +0.031947$ $ASA = +0.86'$ $\rho_2 = -0.0628623$

$R_2 = -15.908$ $s_2' = +4.000$ $SA = +0.0160$

Case II: $s_1 = -2f$ hence, $\sigma_1 = -\frac{\phi}{2}$

Substitute these values into equations (596), (584), (591), (601), and (602), and find:

$\rho_1 = -\rho_2 = +0.241779$ $\sigma_2' = +0.125$ $ASA = +0.90'$

$A = +0.03364$ $R_1 = -R_2 = +4.136$ $s_2' = +8.000$ $SA = +0.0672$

Case III: $s_1 = -f$ hence $\sigma_1 = -\phi$

Substitute these values into equations (597), (582), (584), (598), and (601), and find:

$\rho_1 = +0.062863$ $R_1 = +15.908$ $\sigma_2' = \pm 0$

$ASA = +0.86'$ $\rho_2 = -0.4206965$ $R_2 = -2.377$

$s_2' = \pm \infty$ $A_{Min} = +0.03195$ $SA = +0.04$ diopters

In the latter case the ASA is equal to ${}_r\alpha_2$ for $h_1 = +0.25$

$${}_rs_2' = -1018.82 \text{ inches and hence, } SA = \frac{39.37}{1018.82} = +0.04 \text{ diopters}$$

The three lenses just computed are shown in figure 155. At the bottom of the diagrams the spherical aberration for the lenses is shown graphically. As ordinates, we plot the height of incidence h_1, and as abscissas, the corresponding SA values. The abscissa scales are plotted as five times the ordinate scale.

Example 5:

A plane-convex lens has a focal length of 4 inches. What must be the refractive index of the glass if the lens will produce minimum spherical aberration when the object is at infinity?

We have: $f = 4.00$ $\phi = 0.25$ $\sigma_1 = 0$ $h_1 = 0.25$ $\alpha_o = -1°30'$

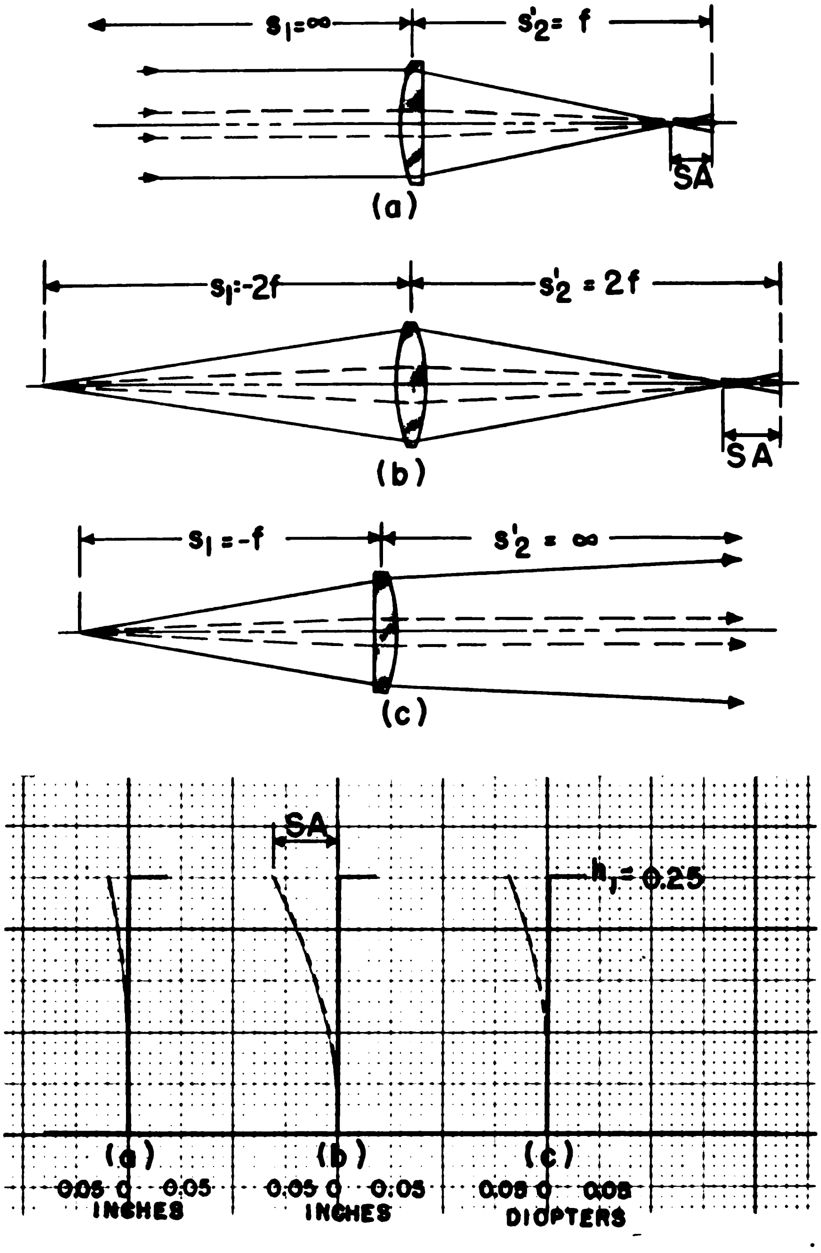


Figure 155. Single thin lenses with minimum spherical aberration for various object distances.

With the aid of equation (599) we find n equals 1.68619; substituting these values into equations (595), (582), (584), (598), (601), and (602), we find:

$\rho_1 = +0.364344$ $\quad R_1 = +2.7447$ $\quad \sigma_2' = +0.25$

$A = +0.02180$ $\quad ASA = +0.59'$ $\quad \rho_2 = \pm 0$

$R_2 = \pm \infty$ $\quad s_2' = +4.000$ $\quad SA = +0.0109$

Example 6:

Compute the radii of a single thin lens with minimum spherical aberration when the object is at infinity using different types of glass.

We have: $f = 4.00$ $\quad \phi = 0.25$ $\quad \sigma_1 = 0$ $\quad h_1 = 0.25$ $\quad \alpha_0 = -1°30'$

For each type of glass we apply successively equations (595), (582), (598), (601), and (602). The results are tabulated below:

	a	b	c	d	e
n =	1.4645	1.5725	1.6170	1.7200	1.9229
R_1 =	+ 2.2374	+ 2.5103	+ 2.6077	+ 2.8058	+ 3.1084
R_2 =	— 10.9569	— 26.0967	— 46.0596	+ 108.8732	+ 19.6738
A =	0.0372	0.0278	0.0251	0.0205	0.0150
ASA =	1.00′	0.75′	0.67′	0.55′	0.40′
SA =	0.0186	0.0139	0.0126	0.0103	0.0075

The lenses and the graphical spherical aberration errors are shown in figure 156.

Example 7:

A single thin collective lens has a focal length of 4 inches and the refractive index is 1.5170. What are the object distances for which the spherical aberration is eliminated?

We have: $f = 4.00$ $\quad \phi = 0.25$ $\quad n = 1.5170$ $\quad h_1 = 0.25$ $\quad \alpha_0 = -1°30'$

The object distances are computed with equation (600) and we find:

$$s_1 = +2.306974 \quad \text{or} \quad s_1 = -1.463126$$

$$\sigma_1 = +0.433468 \quad \text{or} \quad \sigma_1 = -0.683468$$

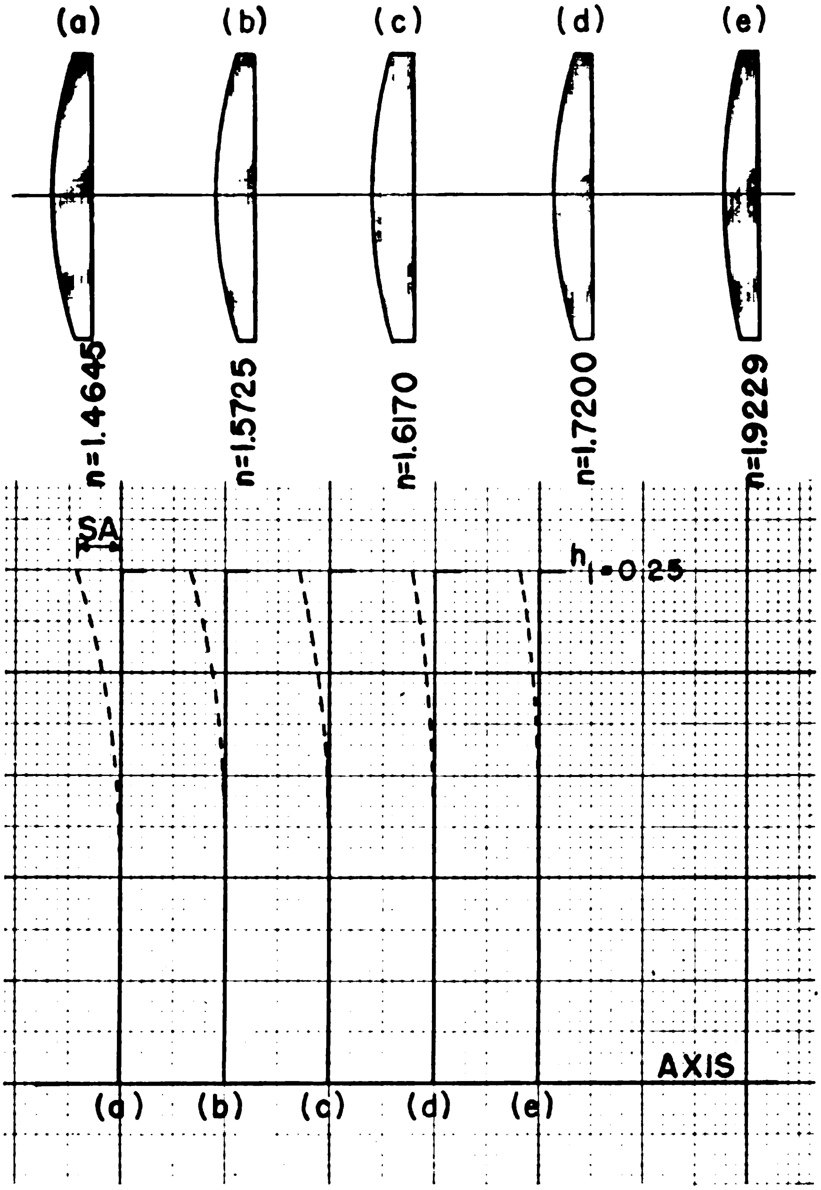


Figure 156. Single thin lenses with minimum spherical aberration for various n values. ($s_1 = \infty$)

Substitute these values into equations (592), (582), (584), (591), (601), and (602). The results are given here:

$\sigma_1 =$	+ 0.43347	— 0.68347
$R_1 =$	+ 0.9605	— 1.7935
$R_2 =$	+ 1.7935	— 0.9605
$s_2' =$	+ 1.4631	— 2.307
A =	± 0	± 0
ASA =	± 0	± 0
SA =	± 0	± 0

The lenses are illustrated in figure 157.

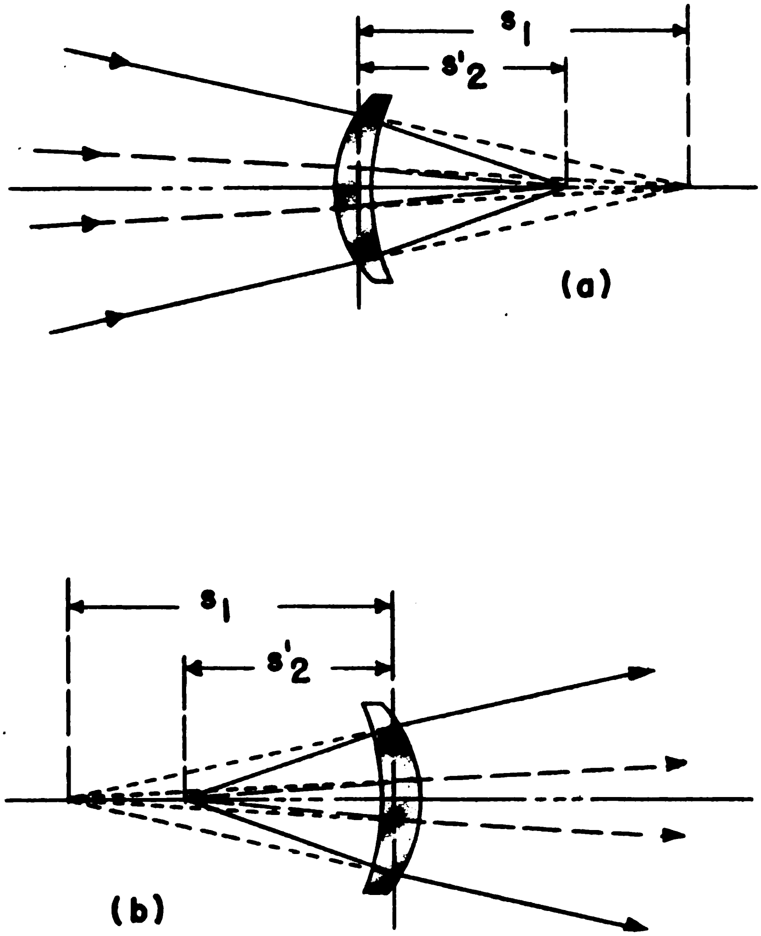


Figure 157. Single thin lenses free from spherical aberration for definite object distances.

Example 8:

A single thin lens has a focal length of 4 inches and the refractive index is 1.5170. The object is at infinity. Determine the radii of the lens.

We have: $f = 4.00$ $\phi = 0.25$ $n = 1.5170$ $h_1 = 0.25$ $\alpha_0 = -1°30'$

Substituting these values into equations (607), (582), (584), (591), (605), (601), (602), (616), and (617), we find:

$\rho_1 = +0.442117$ $R_1 = +2.2618$ $\sigma_2' = +0.25$

$A_{\text{Min}} = +0.032213$ $ASA = +0.87'$

$\rho_2 = -0.0414412$ $R_2 = -24.1301$ $s_2' = +4.00$ $B = \pm 0$

$AC = \pm 0$

$SA = +0.0161$ $\text{Coma} = \pm 0$

This lens is shown in figure 158.

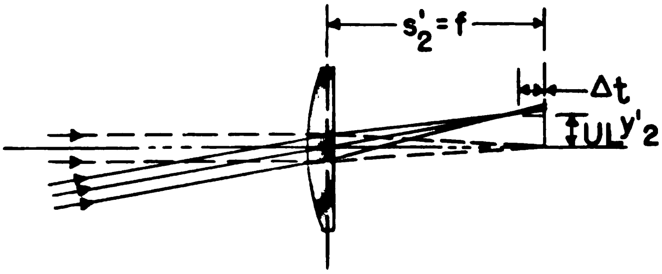


Figure 158. The lens is corrected for coma ($s_1 = \infty$)

Example 9:

Compute the constants of a single thin collective lens of 4 inch focal length and refractive index of 1.517 so that the image is free from coma when the object is 8 inches in front of the lens.

We have: $f = 4.00$ $\phi = 0.25$ $\sigma_1 = -0.125,$ $h_1 = 0.25$ $\alpha_0 = -1°30'$

Applying successively equations (596), (591), (601), (602), (616), and (617), we find:

$\rho_1 = -\rho_2 = +0.241779$ $s_2' = +8.00$ $A = +0.03363$ $ASA = +0.90'$

$R_1 = -R_2 = +4.136$ $B = \pm 0$ $SA = +0.06726$ $\text{Coma} = \pm 0$

This lens is shown in figure 159.

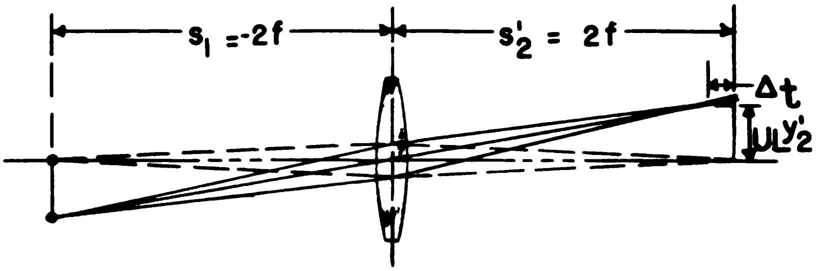


Figure 159. The lens is corrected for coma ($s_1 = -2f$)

Example 10:

The lens computed in example 4(a) has the minimum spherical aberration. Compute the position of a diaphragm so that the coma is eliminated for an object at infinity. We have:

$\rho_1 = +0.420696$ $R_1 = +2.377$ $A = 0.031947$ $h_1 = 0.25$

$\rho_2 = -0.062863$ $R_2 = -15.908$ $SA = 0.0160$ $\alpha_0 = -1°30'$

With the aid of equation (605) we find that B equals -0.008886. Since σ_1 equals 0 then,

$$\xi_1 = \frac{A}{B} = -3.595205 \text{ and } x_1 = -0.278148$$

$$\text{Coma} = A + (\sigma_1 - \xi_1)B = \pm 0$$

This case is illustrated in figure 160.

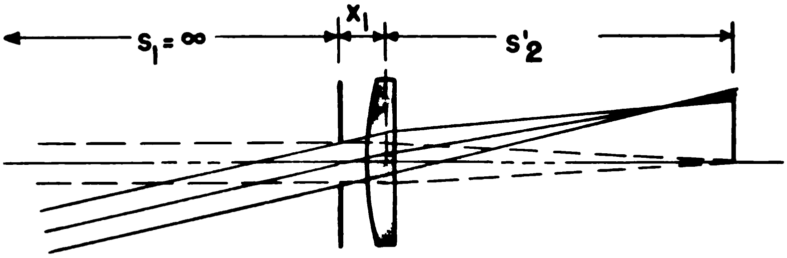


Figure 160. The thin lens with minimum spherical aberration but no coma ($s_1 = \infty$)

Example 11:

Compute the radii, the object, and the diaphragm distances for a single thin lens so that the image is free from spherical aberration and coma.

We have: $f = 4.00$ $\phi = 0.25$ $n = 1.5170$ $h_1 = 0.25$ $\alpha_0 = -1°30'$

Apply successively equations (613), (584), (612), (614), (615), (591), (605), and (611) and find:

$s_1 = -1.363216$ $s_2' = -2.0680$ $x_1 = -0.975607$ $g_2 = -1.0085$

$R_1 = -1.3632$ $R_2 = -0.8216$ $A = B = \pm 0$ $\text{Coma} = \pm 0$

This lens is shown in figure 161.

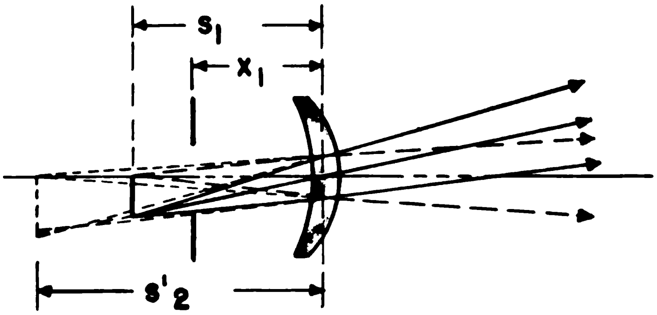


Figure 161. The thin lens free from spherical aberration and coma ($s_1 = R_1$)

Example 12:

A single thin lens whose radii are known is to be corrected for astigmatism when the object is at infinity. Determine the diaphragm distance which will give the desired result. We have:

$\rho_1 = -0.666667$ $R_1 = -1.50$ $f = +4.00$ $\sigma_1 = \pm 0$ $h_1 = +0.125$

$\rho_2 = -1.150226$ $R_2 = -0.8694$ $\phi = +0.25$ $n = 1.5170$ $\alpha_0 = -1°30'$

Apply successively equations (591), (605), and (579), and find:

$A = +0.717239$ $B = -0.459923$ $C = +0.25$ $P = +0.164798$ and $D = \pm 0$

$$S_{IIIa} = x_1^2A - 2x_1B + C = 0 \qquad \text{hence}$$

$$x_1 = \frac{+B \pm \sqrt{B^2 - A\phi}}{A} = -0.390980$$

Determine the errors of this lens by applying equations (591), (605), (579), (581), (580), (624), (625), and (626) and find:

$A = +0.717239$ $S_I = +0.717239$ $ASA = +2.4'$ $SA = +0.0897$

$B = -0.459923$ $S_{II} = -0.179497$ $AC = \pm 0.38'$ $\text{Coma} = \pm 0.00044$

$C = +0.25$ $S_{III} = +0.164796$ $APC = -0.05'$ $\Delta t = -0.000904$

$P = +0.1647989$ $S_{IV} = +0.164798$ $ASP = -0.05'$ $\Delta S = -0.000904$

$D = \pm 0$ $S_{IIIa} = \pm 0$ $AD = -0.01'$ $DIST = -0.000014$

$S_{IV} = +0.164796$ $\quad$ $AST = \pm 0$ $\quad$ $S_V = +0.189616$

$FC = -0.000904$ $\quad$ $R_s = R_t = -6.068$

The lens is shown in figure 162 and in figure 163 the field curves are shown graphically.

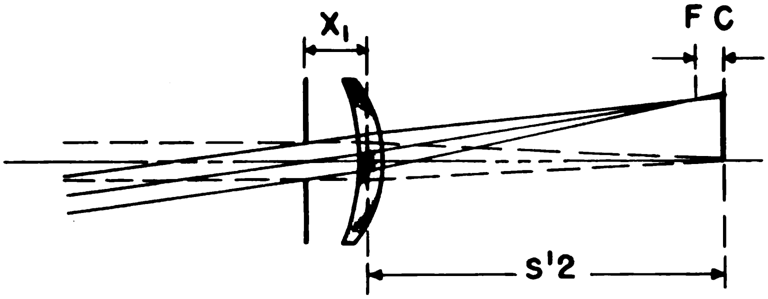


Figure 162. The lens is free from astigmatism.

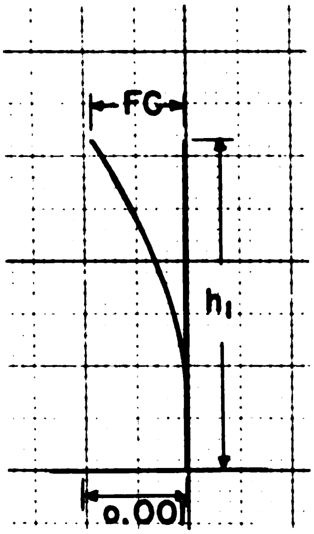


Figure 163. Field curves for lens of example 12.

Example 13:

Compute the radii for a single thin lens so that the mean field curvature is a minimum.

We have: $f = +4.00$ $\quad \phi = +0.25$ $\quad n = 1.5170$ $\quad \sigma_1 = \pm 0$ $\quad h_1 = +0.125$ $\quad \alpha_0 = -1°30'$

Apply equations (620), (621), (622), and (623) and find:

$g_3 = -2.475618$ $\quad x_1 = -0.278130$ $\quad \rho_1 = -2.152436$ $\quad \rho_2 = -2.635995$

$S_{III} = +0.016535$ $\quad R_1 = -0.464590$ $\quad R_2 = -0.379363$

Apply equations (589), (605), (579), (581), (580), (624), (625), and (626) and find:

$A = +\,3.869470$ $S_{I} = +\,3.869470$ $ASA = +\,13'$ $SA = +\,0.484$

$B = -\,1.076217$ $S_{II} = -\,0.000001$ $AC = \pm\,0$ $\text{Coma} = \pm\,0$

$C = +\,0.25$ $S_{III} = +\,0.016811$ $APC = -\,0.02'$ $\Delta t = -\,0.0003628$

$D = \pm\,0$ $S_{IV} = +\,0.115469$ $ASP = -\,0.03'$ $\Delta S = -\,0.000633$

$P = +\,0.164798$ $S_{IIIa} = -\,0.049328$ $AD = -\,0.005'$ $DIST = -\,0.000$

$S_{IVa} = +\,0.066140$ $AST = -\,0.0002752$ $S_{V} = +\,0.087929$

$FC = -\,0.0005224$ $R_{s} = -\,8.660$ $R_{t} = -\,59.485$

This lens is shown in figure 164 and in figure 165 the field curves are shown graphically.

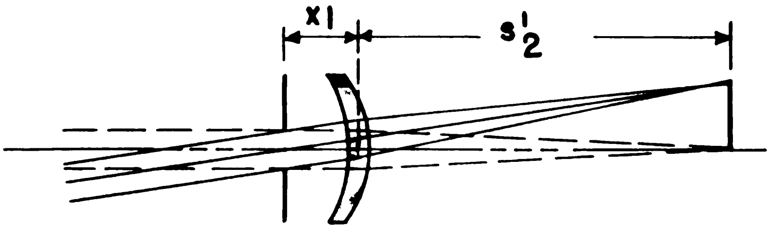


Figure 164. The lens has the minimum field curvature.

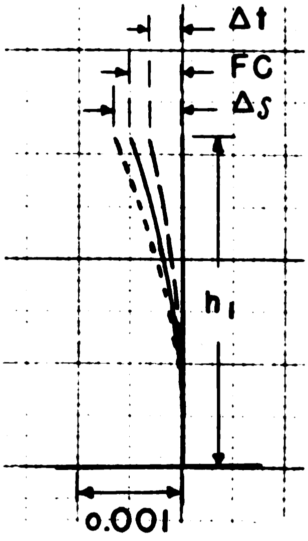


Figure 165. Field curves for lens of example 13.

The distortion is negligible in a single thin lens. The cause of distortion is illustrated in figure 166. The ideal image height, y', equals $-f \tan \alpha_0$. If the principal ray intersects the image at the same height y' there will be no distortion and the image will appear in its normal form as shown in figure 167(b). In figure 166 the principal ray intersects the image plane at a smaller height and the image is distorted. Hence, $y_2' - y'$ is negative and the image will be barrel-shaped as shown in figure 167(a). If $y_2' - y'$ is positive the image will be cushion-shaped as illustrated in figure 167(c).

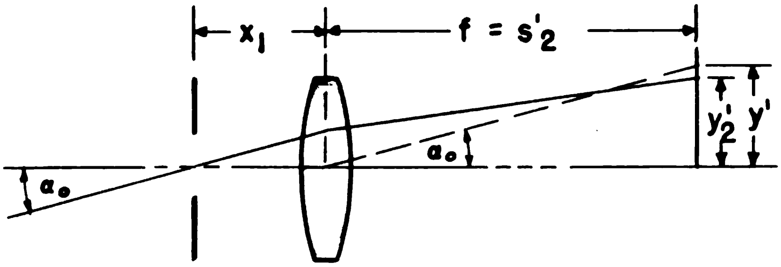
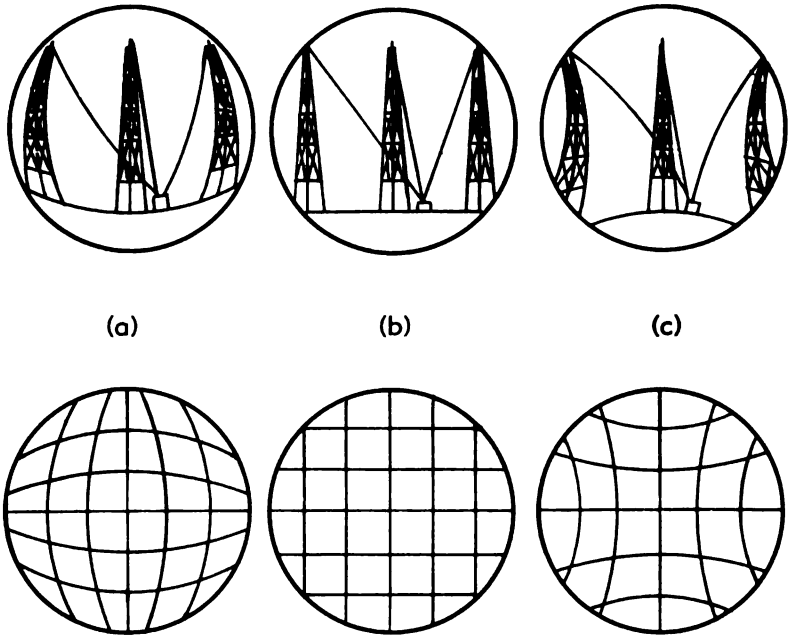


Figure 166. Lens with barrel distortion.



(a). Image has barrel, or negative distortion.
(b). Image is free from distortion.
(c). Image has cushion, or positive distortion.

Figure 167. Images formed by a lens.

c. The single thick lens

In the case of thick lenses the Seidel sum coefficients are written as follows:

$$A_v = H_v^4 Q_v^2 \Delta \left(\frac{\sigma}{n}\right)_v \tag{627}$$

$$\epsilon_v = \frac{1}{H_v^2 Q_v} \tag{628}$$

$$\delta_v = \frac{d_{v-1}}{n_{v-1} H_{v-1} H_v} \qquad (\text{note: } d_0 = 0) \tag{629}$$

$$\tau_v = \epsilon_v + \delta_v \tag{630}$$

$$H_1 = 1 \quad H_v = (1 - d_{v-1} \sigma'_{v-1}) H_{v-1} \tag{631}$$

The values of Q_v, $\Delta\left(\frac{\sigma}{n}\right)_v$, $\Delta\left(\frac{1}{n}\right)_v$, B_v, C_v, D_v and P_v are computed with equations (573), (574), (576), (578), (579), (580), and (581).

The values thus computed are now substituted into the Seidel sum equations.

We proceed to compute the radii of a single thick lens so that the astigmatism and the field curvature are eliminated, that is, $S_{III} = S_{IIIa} = 0$. After numerous substitutions and reductions we derive the following equations:

$$K = 1 - \frac{n-1}{n(z-1)} \left\{ 1 \pm \frac{1}{\sqrt{z\left\{n^2(z-1)+1\right\}}} \right\} \tag{632}$$

$$R_1 = R_2 = (n-1)(1-z)f \tag{633}$$

$$d = \frac{nR_1(1-z)}{n-1} \tag{634}$$

$$x_1 = -\frac{KR_1}{n-K} \tag{635}$$

Example 14:

A single thick lens has a focal length of 4 inches and the refractive index of the glass is 1.5170. What are the radii and the diaphragm distance if the lens is to be free from astigmatism and field curvature?

We have: $f = +4$ $\quad \sigma_1 = \pm 0$ $\quad n = 1.5170$ $\quad S_{III} = S_{IVn} = \pm 0.$

We first determine the value of K by assuming various values for z. In order that the discriminant in equation (632) does not have imaginary roots, the value of z must not lie between 0 and $\left(1-\frac{1}{n^2}\right)$, that is, in our example, it must not lie between 0 and 0.565.

When $z = 1$ then $R_1 = R_2 = d = x_1 = 0$. Below are tabulated the constants of the lens for various values of z.

	$z =$	$+1.00$	$+1.05$	$+1.10$	$+1.20$	$+1.25$
(a)	$K =$		$+0.48318$	$+0.52173$	$+0.58325$	$+0.60824$
	$R_1 = R_2 =$	± 0	-0.1034	-0.2068	-0.4136	-0.5170
	$d =$	± 0	$+0.01517$	$+0.06068$	$+0.24272$	$+0.37925$
	$x_1 =$	± 0	$+0.04833$	$+0.10841$	$+0.25835$	$+0.34604$
(b)	$K =$		-12.11535	-5.33781	-1.99129	-1.33468
	$x_1 =$	± 0	-2.29734	-0.16103	-0.23476	-0.24197

The lens with the constants given in the last column ($x_1 = -0.24197$) is shown in figure 168.

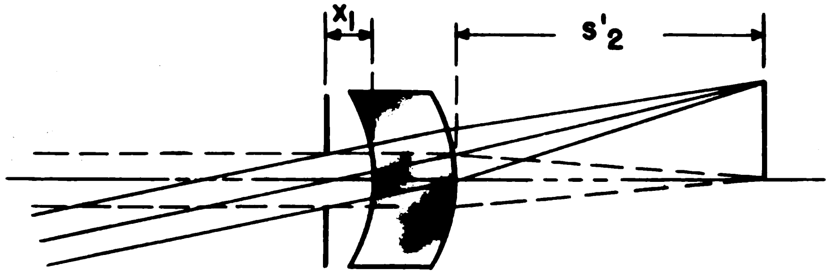


Figure 168. A single thick lens free from astigmatism and field curvature.

Substituting these constants into the equations (627), (628), (629), (630), (631), and (624) we find:

$S_I = +4.4642$ $S_{III_a} = \pm 0$ $S_{II} = +1.3125$ $S_{IV_a} = \pm 0$

$S_{III} = \pm 0$ $S_V = +0.199772$ $S_{IV} = \pm 0$

When $h_1 = +0.25$ and $\alpha_0 = -1°30'$ we find the linear errors with the aid of formulae (626) as follows:

$\text{SA} = +3.4877$ $\Delta S = \Delta t = \text{AST} = \text{FC} = \pm 0$

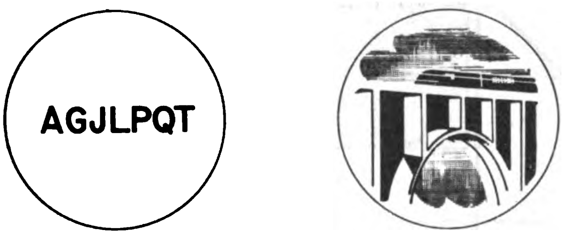
$\text{Coma} = +0.0161$ $\text{DIST} = -0.00002$

SECTION IX

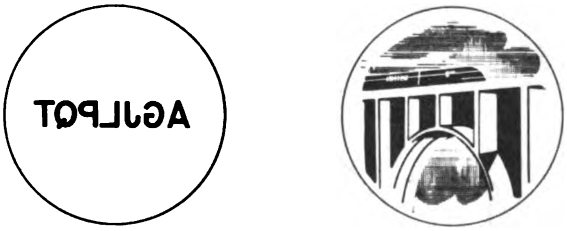
PRISMS AND PRISM SYSTEMS

101. Introduction

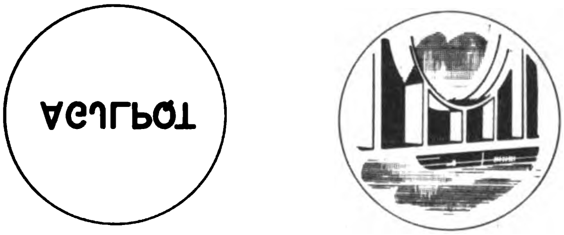
In this section are presented construction details for different prisms that are used in fire control instruments. Dimensions are given in terms of the diameter (A) of a cylindrical beam of light which can be transmitted by the prism or prism system. For a general treatment of prisms, see section IV.



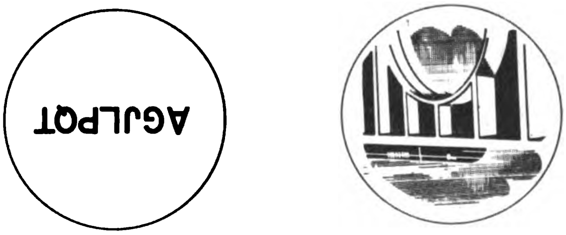
NORMAL IMAGES



REVERTED IMAGES



INVERTED IMAGES



INVERTED-REVERTED IMAGES

Figure 169. Inversion and reversion.

102. Porro Prism System

In 1850 the Italian engineer Porro designed the prism system discussed here. This system consists of two right-angle prisms, usually identical in construction, placed at right angles to each other. It is a direct vision prism system but the axis is displaced by the amount Δ. This system will invert and revert the image.

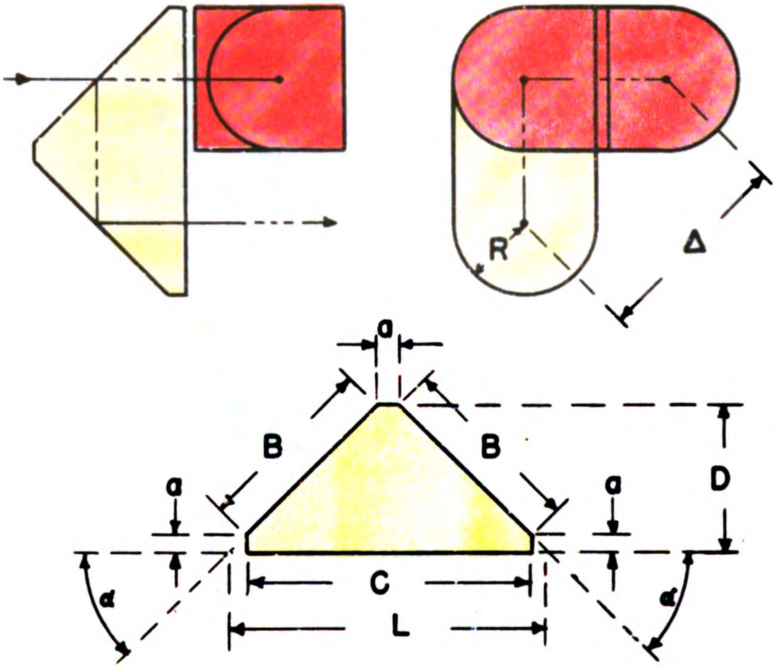


Figure 170. Porro prism system.

$A = 1.00$ $n = 1.5170$ $\alpha = 45°$ (These values are given)

$a = 0.10$ (chosen arbitrarily) $R = A/2 = 0.50$

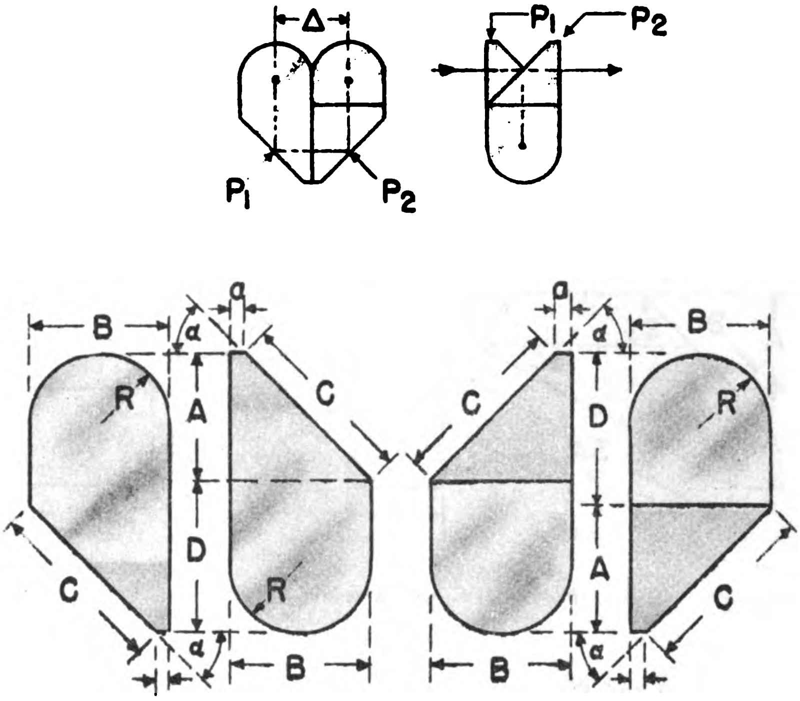
$B = 1.4142A = 1.4142$ $C = 2A + a = 2.1$ $D = A + a = 1.1$

$L = 2A + 3a = 2.30$ $\Delta = 1.4142\ (A + a) = 1.5556$

$d = 2\ (2A + 3a) = 4.60$ $d/n = 3.0324$

103. Abbe's Modification of the Porro Prism System

This prism system consists of two prisms cemented together. It will invert and revert the image. The system is a direct vision prism but the line of sight will be displaced by the amount Δ.



Prism No. 1. *Prism No. 2.*

Figure 171. Abbe prism system.

$A = 1.00$ $n = 1.5170$ $\alpha = 45°$ $a = 0.10$ (chosen arbitrarily)

$B = A + a = 1.10$ $C = 1.4142A = 1.4142$ $D = A + 2a = 1.20$

$R = B/2 = 0.55$ $\Delta = B = 1.10$

$d = 2\,(2A + 3a) = 4.60$ $d/n = 3.0323$

104. Abbe Prism, Type A

This prism inverts and reverts the image, but will not displace the line of sight; hence, it is a *"Direct Vision Prism."* The prism is made in two pieces which are cemented together.

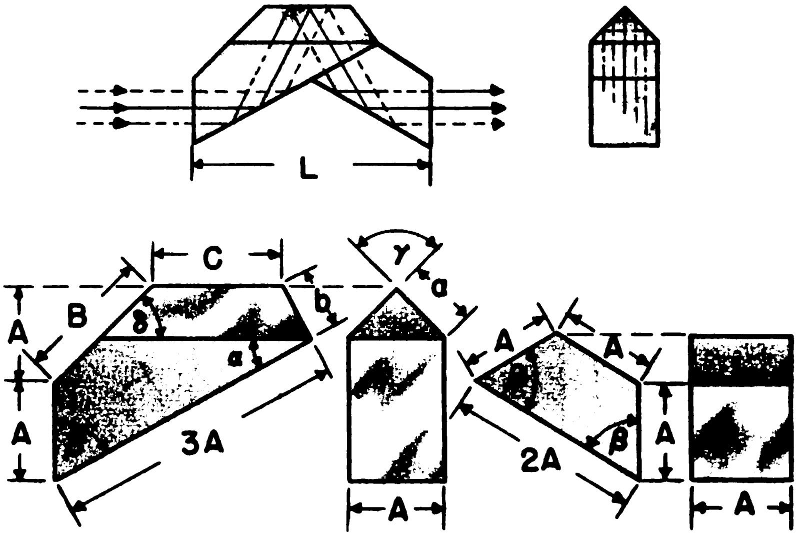


Figure 172. Abbe prism, type A.

$A = 1.00$ $\alpha = 30°$ $\gamma = 90°$

$n = 1.5170$ $\beta = 60°$ $\delta = 45°$

$B = 1.4142A = 1.4142$ $C = 1.3094A = 1.3094$

$a = 0.7071A = 0.7071$ $b = 0.5774A = 0.5774$

$L = 3.4644A = 3.4644$

$d = 5.1962A = 5.1962$ $d/n = 3.4253$

105. Abbe Prism, Type B

This prism is made of three single units which are cemented together. This prism will invert and revert the image but will not deviate the line of sight. It is also a *"Direct Vision Prism."*

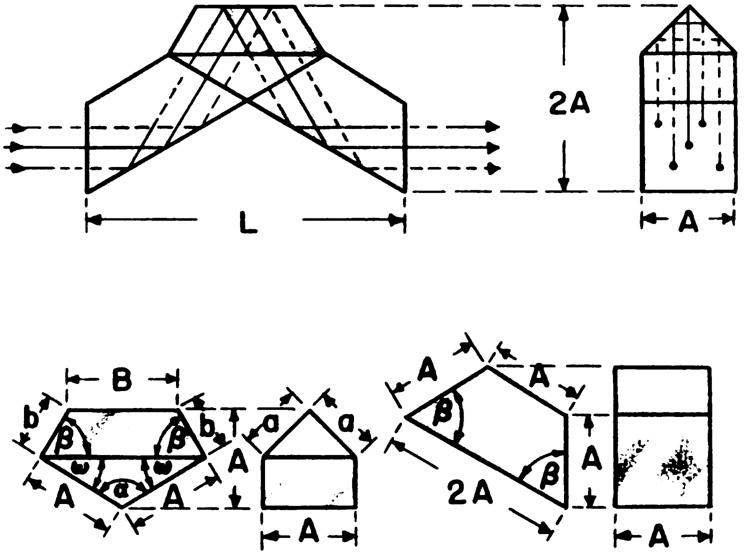


Figure 173. Abbe prism, type B.

A = 1.00 α = 135° γ = 45°

n = 1.5170 β = 60° ω = 30°

a = 0.7071A = 0.7071

b = 0.5773A = 0.5773

B = 1.1547A = 1.1547

L = 3.4641A = 3.4641

d = 5.1962A = 5.1962

d/n = 3.4253

106. Leman Prism

The Leman prism will revert and invert the image. The line of sight will be displaced (laterally by an amount equal to 3A inches).

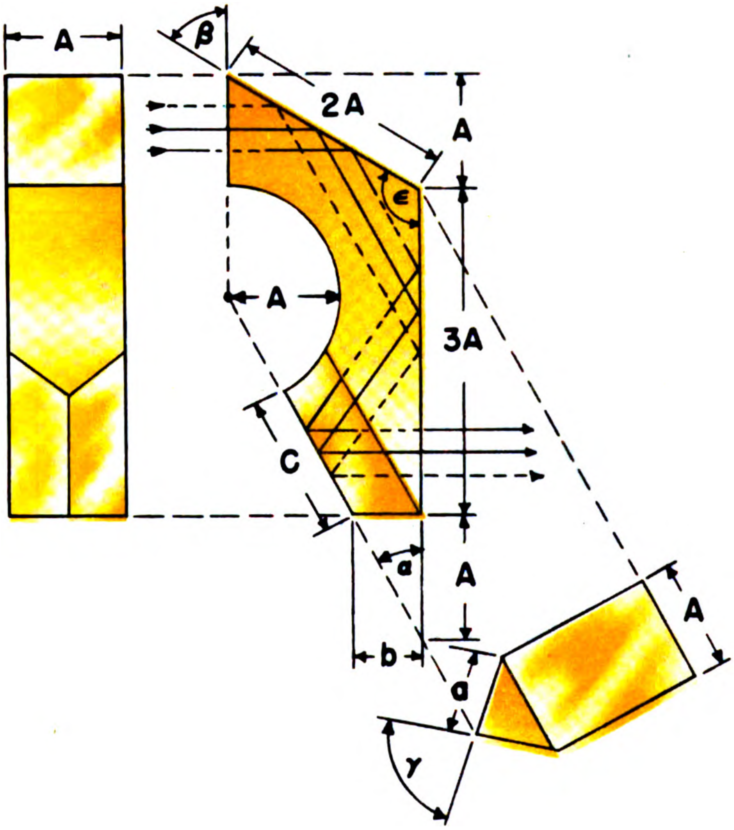


Figure 174. Leman prism.

$A = 1.00$
$n = 1.5170$
$\alpha = 30°$
$\beta = 60°$
$\gamma = 90°$
$\varepsilon = 120°$
$d = 5.1962A = 5.1962$

$B = 1.7321A = 1.7321$
$a = 0.7071A = 0.7071$
$C = 1.3099A = 1.3099$
$b = 0.5774A = 0.5774$
$d/n = 3.4253$

107. Amici Prism

During his life, 1784 to 1863, the Italian astronomer Amici designed many prisms. This is one of them. This prism will revert and invert the image and, at the same time, it will deviate the line of sight through an angle of 90°.

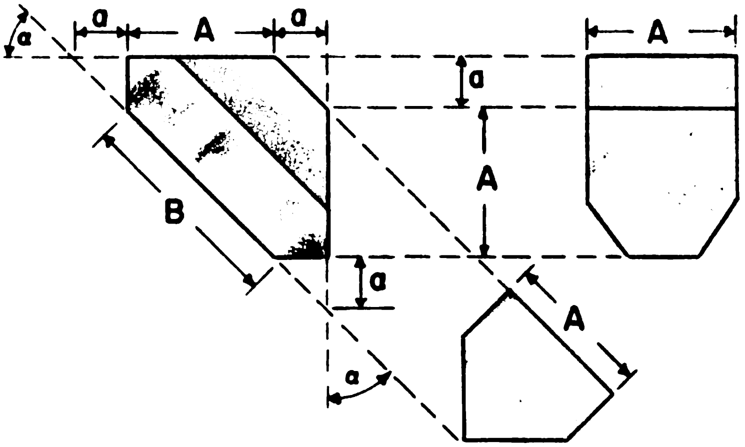


Figure 175. Amici prism.

$A = 1.00$
$n = 1.5170$
$\alpha = 45^\circ$
$B = 1.4142A = 1.4142$
$a = 0.3536A = 0.3536$
$d = 1.7071A = 1.7071$
$d/n = 1.1253$

108. Schmidt Prism

This prism will revert and invert the image and, at the same time, it will deviate the line of sight through an angle of 45°.

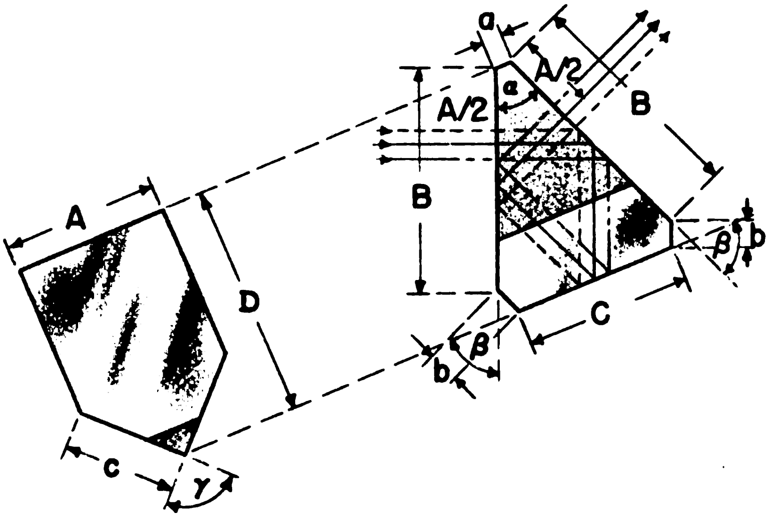


Figure 176. Schmidt prism.

$A = 1.00$
$a = 0.10$ (chosen at will)
$n = 1.5170$
$\alpha = 45^\circ$
$\beta = 67^\circ 30'$

$\gamma = 90^\circ$
$B = 1.4142A + 0.5412a = 1.4683$
$C = 1.0824A = 1.082$
$D = 1.4142A + 2.3890a = 1.6531$

$b = 1.8478a = 0.1848$
$c = 0.7071A = 0.7071$
$d = 3.4142A = 3.4142$
$d/n = 2.2506$

109. Right-Angle Prism

This single prism will deviate the line of sight through an angle of 90°. The image will be inverted when the prism is held before the eye as shown in figure 177(a), and it will appear reverted when the prism is turned through an angle of 90° as illustrated in figure 177(b).

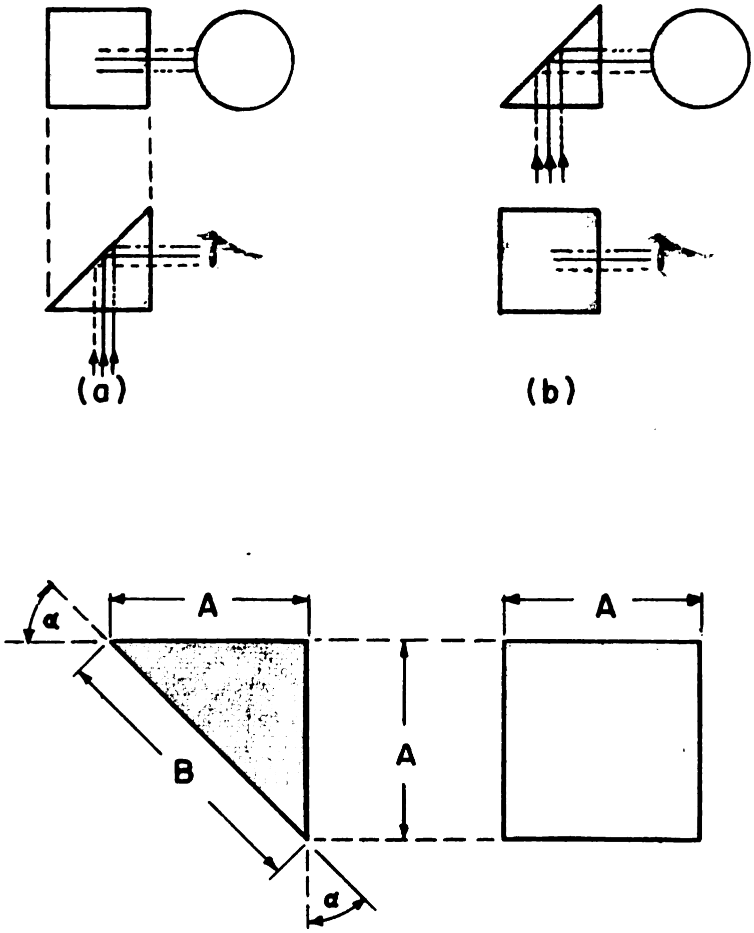


Figure 177. Right-angle prism.

$A = 1.00$ $n = 1.5170$ $\alpha = 45°$

$B = 1.4142A = 1.4142$

$d = A = 1.00$ $d/n = 0.6592$

110. Harting-Dove Prism

This direct vision prism is made in one piece. The image will be reverted when the prism is held as shown in figure 178(a), and it is inverted when the prism is turned about the axis through an angle of 90°. It can be used only in parallel light.

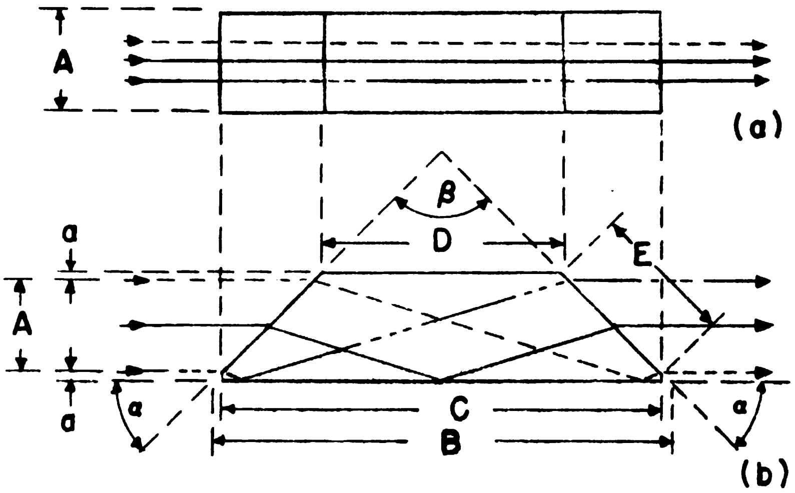


Figure 178. Dove prism.

$A = 1.00 \qquad a = 0.05 \qquad n = 1.5170 \qquad \alpha = 45^\circ \qquad \beta = 90^\circ$

$$B = (A + 2a)\left\{\frac{\sqrt{n^2 - \sin^2\alpha} + \sin\alpha}{\sqrt{n^2 - \sin^2\alpha} - \sin\alpha} + 1\right\} = 4.2271\,(A + 2a) = 4.6498$$

$$C = B - 2a = 4.5498 \qquad D = B - 2(A + 2a) = 2.4498$$

$$E = \frac{a + A}{\cos\alpha} = 1.4142\,(A + a) = 1.4849$$

$$d = \frac{n\,(A + 2a)}{\sin\alpha\,\{\sqrt{n^2 - \sin^2\alpha} - \sin\alpha\}} = 3.3787\,(A + 2a) = 3.7165$$

$$d/n = 2.4499$$

NOTE: See table XXI.

TABLE XXI.

Effect on the Prism Constants When Different Types of Glasses are Used

$n =$	1.5170	1.5725	1.6170	1.7200
$B =$	4.6498	4.4303	4.2822	4.0072
$C =$	4.5498	4.3303	4.1822	3.9072
$D =$	2.4498	2.2303	2.1822	1.9072
$E =$	1.4849	1.4849	1.4849	1.4849
$d =$	3.7165	3.5071	3.3637	3.1084
$\frac{d}{n} =$	2.4499	2.2303	2.0802	1.8072

111. Double Dove Prism

This twin prism consists of two Harting-Dove prisms. Their reflecting surfaces are silvered and then the two halves are cemented together. This method cuts the length of the single Harting-Dove prism in half. This prism performs the duties of a single Harting-Dove prism, and it too must be placed in parallel light only.

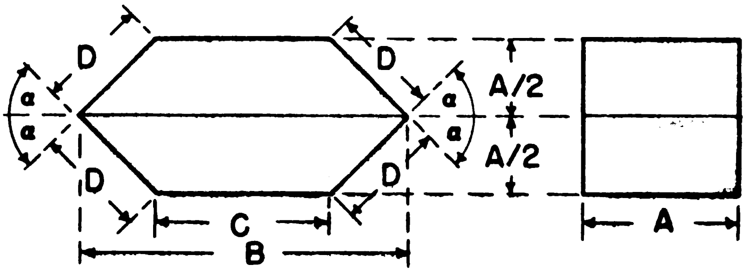


Figure 179. Double Dove prism.

$A = 1.00 \qquad n = 1.5170 \qquad \alpha = 45°$

$$B = \frac{A}{2}\left\{\frac{\sqrt{n^2 - \sin^2\alpha} + \sin\alpha}{\sqrt{n^2 - \sin^2\alpha} - \sin\alpha} + 1\right\} = 2.1136A = 2.1136$$

$$C = B - A = 1.1136 \qquad D = \frac{A}{2\cos\alpha} = 0.7071A = 0.7071$$

$$d = \frac{nA}{2\sin\alpha\,\{\sqrt{n^2 - \sin^2\alpha} - \sin\alpha\}} = nAC = 1.6893$$

$$d/n = 1.1135$$

112. Pechan Prism

This prism performs the same duties as the Harting-Dove prism but it has one great advantage over the latter inasmuch as it may be placed in convergent or divergent light. This will permit the reduction in length or height of the instrument. The surfaces marked B are silvered and covered with a protective coating. The unsilvered reflecting surfaces of the prism are separated by a distance of about 0.002 inch.

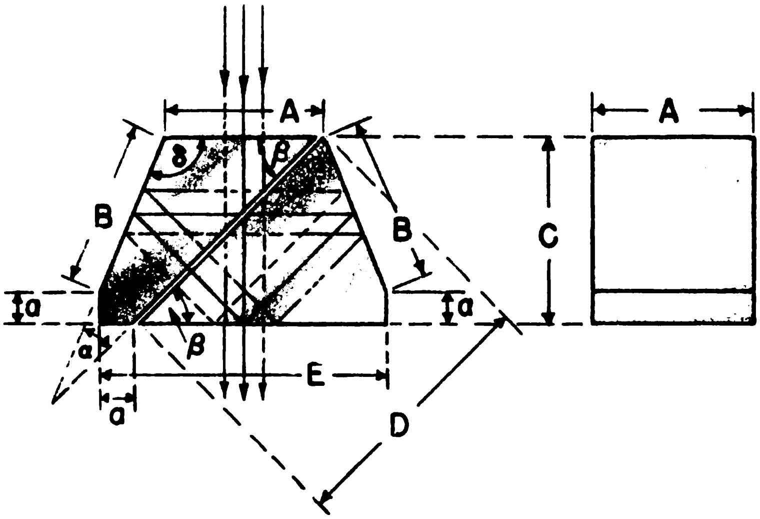


Figure 180. Pechan prism.

$A = 1.00$ $\alpha = 22°30'$ $\gamma = 67°30'$

$n = 1.5170$ $\beta = 45°$ $\delta = 112°30'$

$B = 1.0824A = 1.0824$ $C = 1.2071A = 1.2071$

$a = 0.2071A = 0.2071$ $D = 1.7071A = 1.7071$

$d = 4.6213A = 4.6213$ $d/n = 3.0464$

$E = 1.8284A = 1.8284$

113. Reversion Prism

This prism, which is a modification of the Abbe prism-type A, consists of two elements which are cemented together. Like the Pechan prism, shown in paragraph 112, it may be placed in the path of parallel, converging, or diverging beams of light.

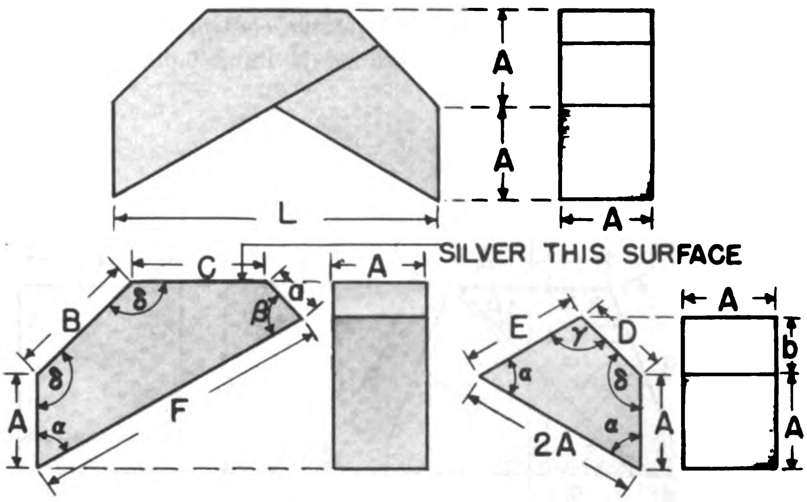


Figure 181. Reversion prism.

A = 1.00
B = 1.4142A
D = 0.8966A
n = 1.5170

C = 1.4641A
E = 1.2679A
d = 5.1962A
α = 60°

β = 75°
γ = 105°
δ = 135°
F = 3.2679A

a = 0.5176A
d/n = 3.4253A
L = 3.4641A
b = 0.6340A

114. Penta Prism

This prism will neither revert nor invert the image but will merely deviate the line of sight through an angle of 90°. The surfaces marked C in figure 182 must be silvered and covered with a protective coating.

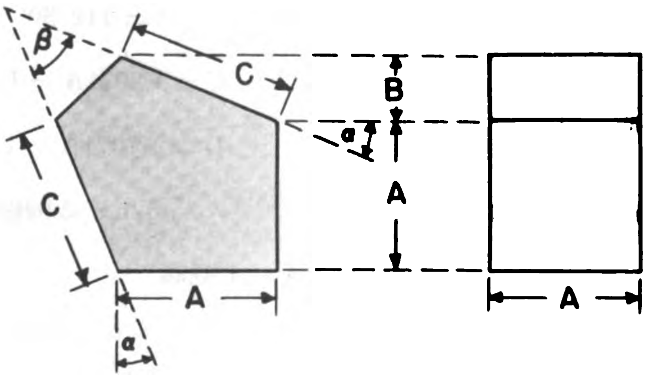


Figure 182. Penta prism.

A = 1.00
n = 1.5170
B = 0.4142A = 0.4142
d = 3.4142A = 3.4142

α = 22°30′
β = 45°
C = 1.0824A = 1.0824
d/n = 2.2506

115. Wollaston Prism

Between the years of 1766 and 1828, the English scientist W. H. Wollaston designed a prism which has been named after him. It is made in one piece of glass and will neither invert nor revert the image, but it will deviate a beam of light through an angle of 90°. It is not used in military instruments due to its unfavorable shape. However, it is still used in an instrument known as "Camere Lucida," or "Camera Clara," the theory of which is explained here. If the observer's eye is placed right above the upper corner of the prism as shown in figure 183, and a sheet of paper P is placed on the table about 10 inches from the eye, the observer will be able, with the aid of a pen, to trace the image of the object on the paper.

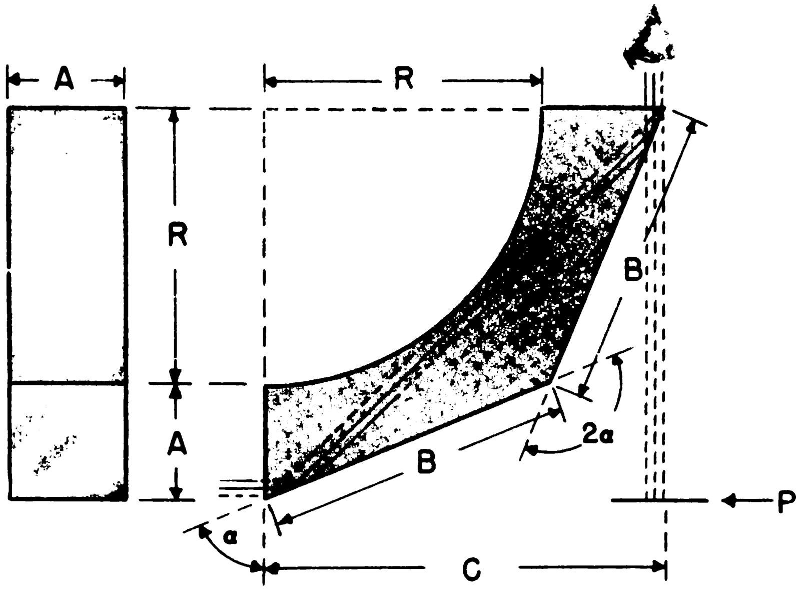


Figure 183. The Wollaston prism.

$A = 1.00$ $n = 1.5170$ $\alpha = 67°30'$

$B = 2.6131A = 2.6131$ $C = 3.4142A = 3.4142$

$R = 2.4142A = 2.4142$

$d = 2R = 4.8284A = 4.8284$ $d/n = 3.1829$

116. Carl Zeiss Prism System

This combination consists of three single prisms (see fig. 184). As a rule the objective is placed between P_1 and P_2; however, it may also be placed in front of the objective prism P_1. This system will invert and revert the image but will not deviate the line of sight. The line of sight will be displaced an amount depending on the distance between the prisms P_1 and P_2.

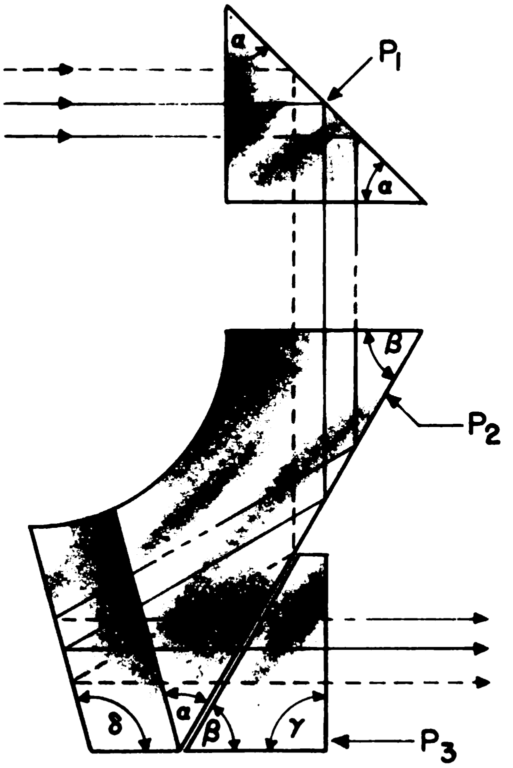


Figure 184. A Carl Zeiss prism system.

$\alpha = 45°$

$\beta = 60°$

$\gamma = 90°$

$\delta = 105°$

117. C. P. Goerz Prism System

This prism system consists of three single prisms as illustrated in figure 185. The light is received by prism P_1, also known as the objective prism. The objective, usually placed between P_1 and P_2 may also be placed in front of P_1. This system will invert and revert the image. The line of sight will not be deviated from its original direction but will be displaced by an amount depending on the distance between the prisms P_1 and P_2.

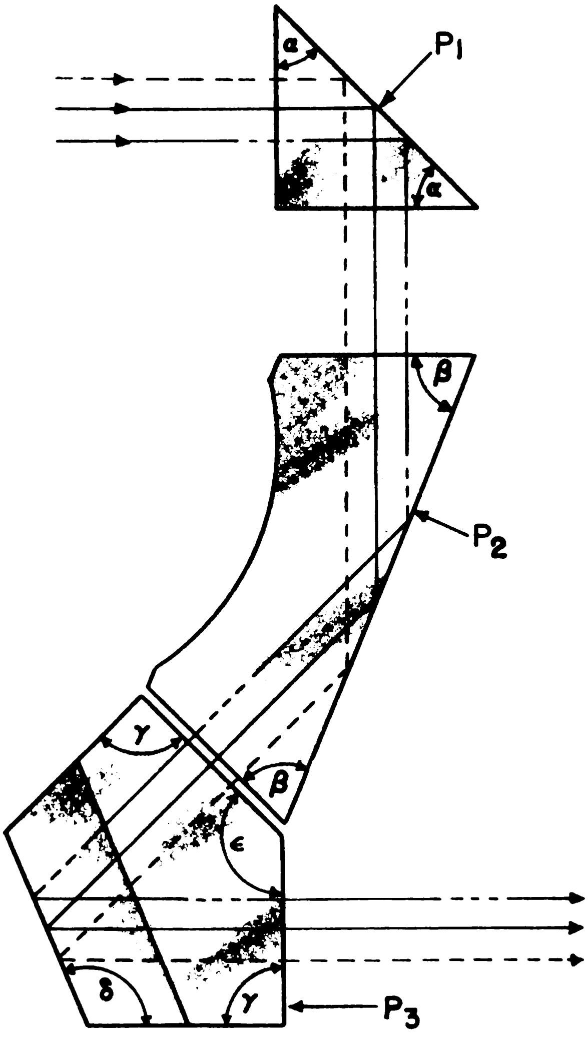


Figure 185. A Goerz prism system.

$\alpha = 45°$
$\beta = 67°30'$
$\gamma = 90°$
$\delta = 112°30'$
$\varepsilon = 135°$

118. Carl Zeiss Ocular Prism

This prism system, used in coincidence type range-finders, is made up of four single prisms, which are cemented together (see fig. 186). Light from the right will enter the system through the rhomboid prism P_1 and, after two internal reflections in this prism, and then three more in P_2 (the last reflection takes place on the silvered portion), the ray will emerge from the prism P_4 and then enter the eye of the observer. The image will be erect but reverted. Light from the left will enter the system through the prism P_3 and, after two internal reflections in this prism it will emerge from the system through P_4 and then it will also enter the eye of the observer. The image will appear inverted and reverted.

In figure 186 the refracting angles of the prism P_2, P_3, and P_4 are 22°30″, and the light is deviated through an angle of 45°. This value may easily be varied by changing the refracting angles of the prisms.

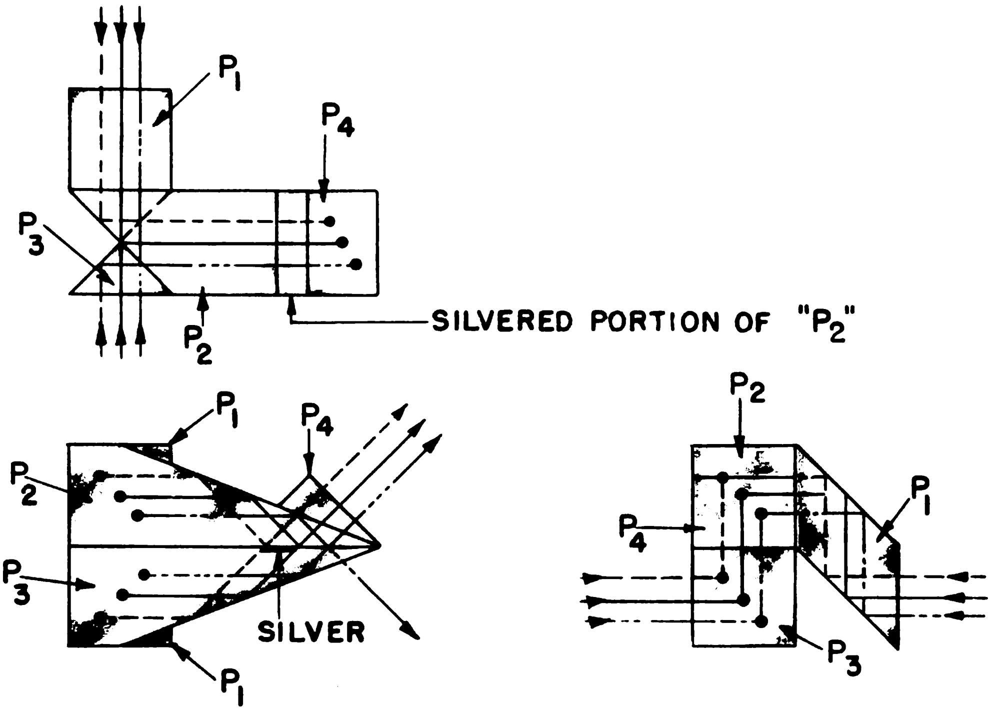


Figure 186. An ocular prism by Zeiss.

119. Barr and Stroud Ocular Prism

This ocular prism system, consisting of four single prisms and a cover, all cemented together, was used during the second world war by Research Enterprise Limited. It has one advantage over the Zeiss prism inasmuch as no silvered surface is required in producing the dividing line between the two images. On the other hand, the production division claims that the cost of manufacturing this prism is about five times that of the Zeiss prism, due to the great difficulties encountered in producing a well defined dividing line.

The prisms P_1, P_2, and P_3 are made of a Boro Silicate crown glass ($n = 1.509$) and the prism P_4 of an Extra Dense flint glass ($n = 1.654$).

The paths through the prism system of the various rays are illustrated in diagrams (a) and (b) of figure 187. The rays of light, after passing through the right objective will enter the prism system through the prism P_1. After a reflection on the hypotenuse of this prism the rays will enter prism P_2, and, after three internal reflections in this

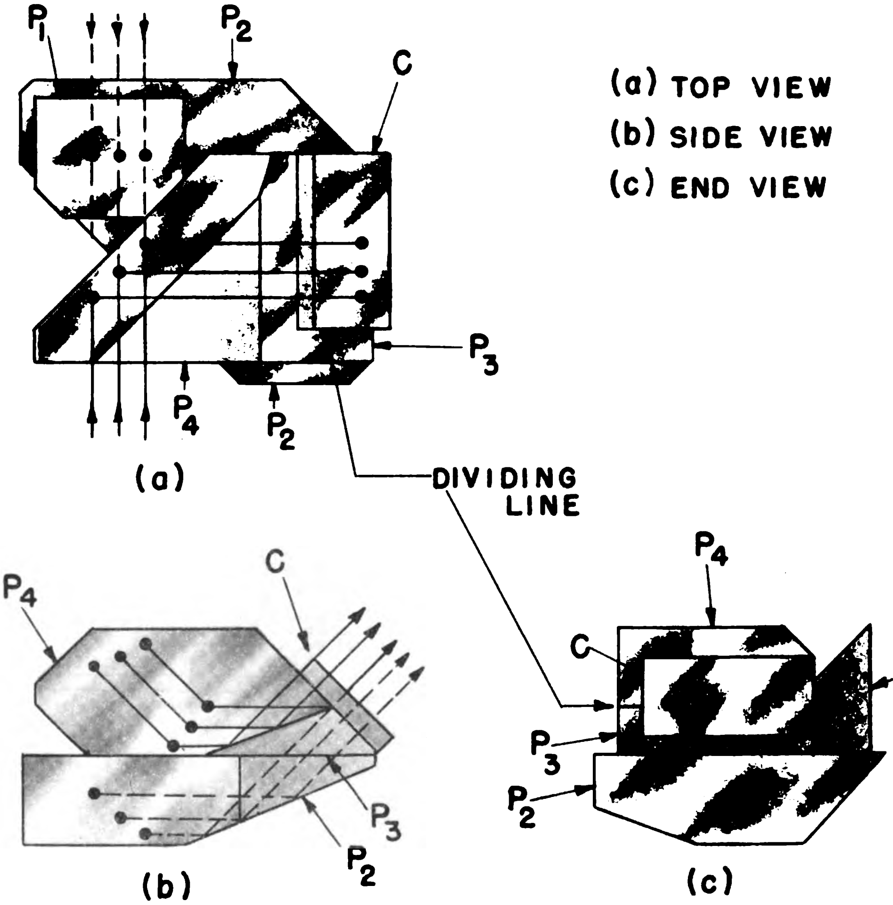


Figure 187. A Barr & Stroud ocular prism.

prism, they will pass undeviated through the prism P_3 and the cover C and will then proceed towards the eye piece. The image seen through this part of the prism system will appear completely inverted.

The rays of light passing through the left objective will enter the prism system through the prism P_4 and will be reflected twice before they reach the dividing line between this prism and prism P_3. Due to the fact that the refractive index of P_4 is much greater than that of prism P_3, the rays will be reflected in an upward direction and emerge from the prism system parallel to the other rays. The image seen through this portion of the prism system will be erect but reverted.

120. Carl Zeiss Coincidence Prism System

This prism system, illustrated in figure 188, consists of two single prisms, P_1 and P_2. The lower half of the upper reflecting surface of P_1 is silvered and then the two prisms are cemented together.

Light from the right will enter first P_1 at the lower entrance surface and, after three internal reflections it will emerge from the system at the upper exit surface. The image will appear reverted.

Light from the left will enter through the prism P_2 and, after three reflections in the prism, it will enter prism P_1 through the unsilvered portion of the reflecting surface. The light will pass through P_1 undeviated before emerging from the prism system. The image will also appear reverted.

This system is used in long base range finders. It is placed between the left objective and its image plane. The images formed by the left and right objective are formed in a plane normal to the line of sight through the dividing line of the silvered and the unsilvered portions of the reflecting surface. A lens erecting system will then transmit these images into the front focal plane of the ocular.

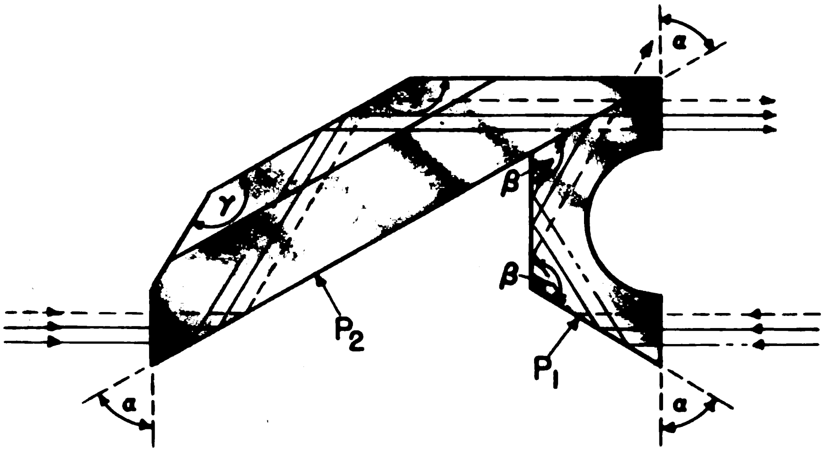


Figure 188. A Zeiss coincidence prism system.

$\alpha = 60°$ $\beta = 120°$ $\gamma = 150°$

121. Carl Zeiss Binocular-Ocular Prism System

This system, illustrated in figure 189, is used in monocular telescopes (or microscopes) when both eyes are to view the image presented by the objective. This system is made up of four single prisms, namely, the right angle prism P_1 cemented to the rhomboid prism R_1; the cemented surface will split the beam of light. The light passing through R_1 and P_1 will, before entering the eye, pass through the prism R_2. The other ray will pass through the block B which has been added to the system to equalize the length of the light-paths in glass. The Interpupillary Distance is designated by the letter D. Its value varies between the limits of $D_m = 58$ mm $= 2.283$ inches and $D_M = 72$ mm $= 2.835$ inches.

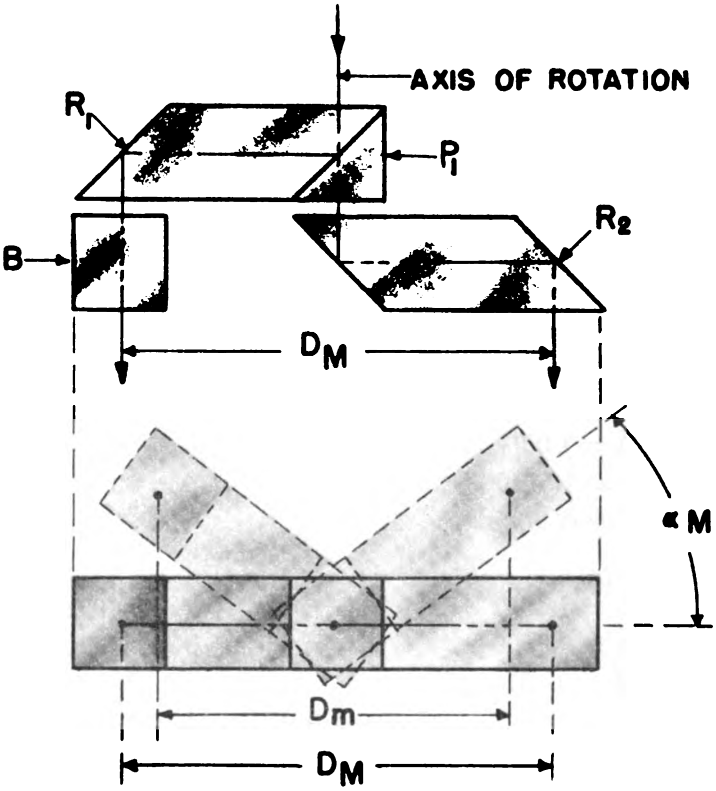


Figure 189. A Zeiss binocular-ocular prism.

$$\cos \alpha_M = \frac{D_m}{D_M} = \frac{2.283}{2.835} = 0.805291$$

$$\alpha_M = 36°22'3''$$

122. Frankford Arsenal Prisms 1 to 7

a. Frankford Arsenal Prism No. 1.

This prism will revert and invert the image and, at the same time, it will deviate the line of sight through an angle of 115°.

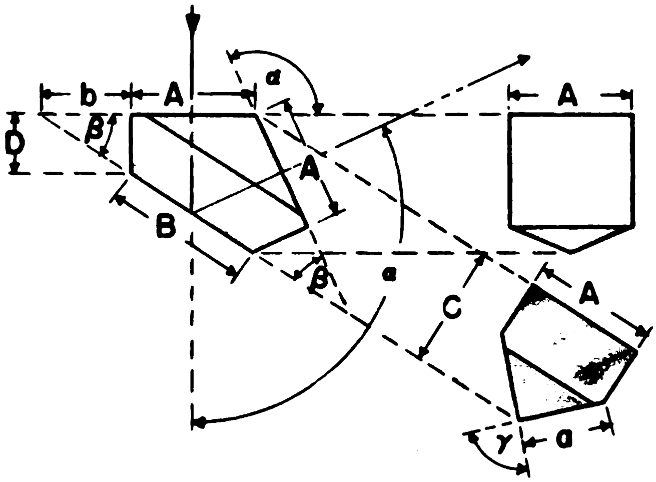


Figure 190. Frankford Arsenal prism No. 1.

$A = 1.00$
$n = 1.5170$

$\alpha = 115°$
$\beta = 32°30'$
$\gamma = 90°$

$B = 1.1857A = 1.1857$
$C = 0.9306A = 0.9306$
$D = 0.4613A = 0.4613$

$a = 0.7071A = 0.7071$
$b = 0.7320A = 0.7320$
$d = 1.5697A = 1.5697$

$d/n = 1.0347$

b. Frankford Arsenal Prism No. 2.

This prism is made in one piece. It will invert and revert the image and, at the same time, it will deviate the line of sight through an angle of $\delta = 60°$.

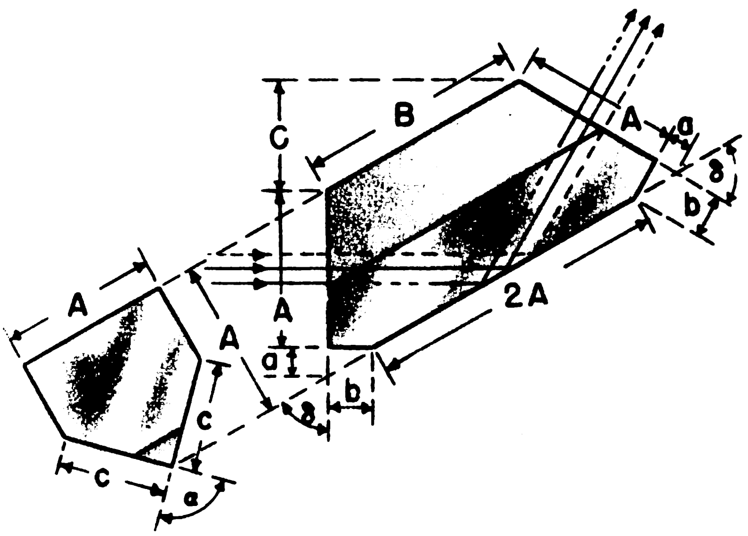


Figure 191. Frankford Arsenal prism No. 2.

$A = 1.00$ $n = 1.5170$ $\delta = 60°$ $\alpha = 90°$

$a = 0.1547A = 0.1547$ $b = 0.2680A = 0.2680$

$B = 1.4641A = 1.4641$ $C = 0.7321A = 0.7321$

$d = 2.2690A = 2.2680$ $d/n = 1.4951$

c. *Frankford Arsenal Prism No. 3.*

This prism is made in one piece. It will deviate the line of sight through an angle of 90° in the horizontal plane and, at the same time, through an angle of 45° in an upward direction. The observer, standing at right angles to the line of sight, will see an inverted and reverted image.

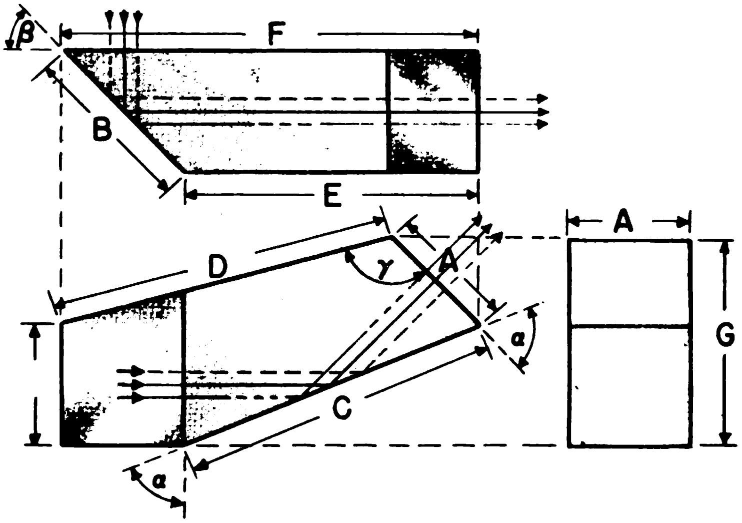


Figure 192. Frankford Arsenal prism No. 3.

$A = 1.00$

$n = 1.5170$

$\alpha = 67°30'$

$\beta = 45°$

$\gamma = 120°21'40''$

$B = 1.4142A = 1.4142$

$C = 2.6131A = 2.6131$

$D = 2.7979A = 2.7979$

$E = 2.4142A = 2.4142$

$F = 3.4142A = 3.4142$

$G = 1.7071A = 1.7071$

$d = 3.4142A = 3.4142$

$d/n = 2.2506$

d. Frankford Arsenal Prism No. 4.

This prism is made of one piece of glass. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 45° in the vertical plane. The observer, standing at right angles to the line of sight, will see the image reverted.

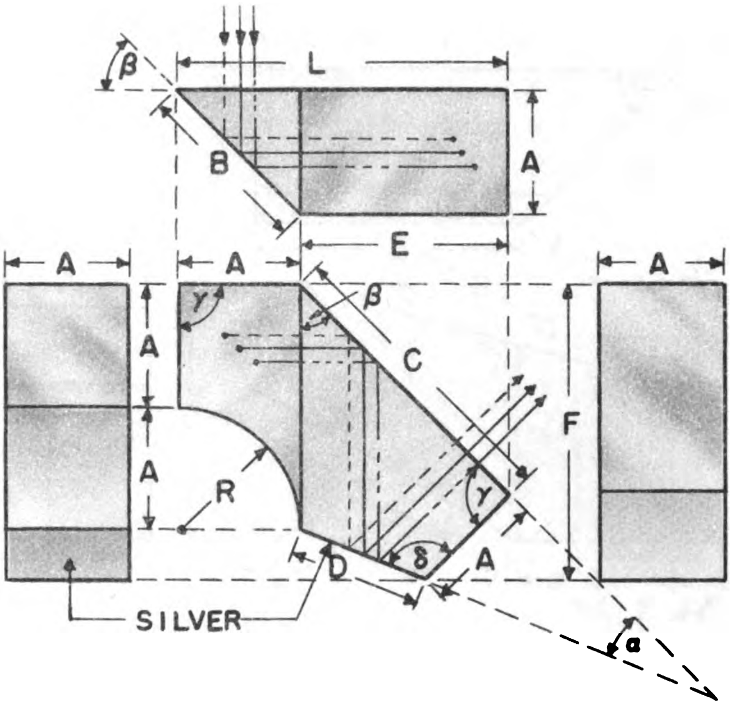


Figure 193. Frankford Arsenal prism No. 4.

$A = 1.00$ $\alpha = 22°30'$ $\gamma = 90°$

$n = 1.5170$ $\beta = 45°$ $\delta = 112°30'$

$B = 1.4142A = 1.4142$ $C = 2.4142A = 2.4142$

$D = 1.0824A = 1.0824$ $E = 1.7071A = 1.7071$

$F = 2.4142A = 2.4142$ $L = 2.7071A = 2.7071$

$R = A = 1.00$

$d = 4.4142A = 4.4142$ $d/n = 2.9098$

e. Frankford Arsenal Prism No. 5.

This prism is made in one piece. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 60° in the vertical plane. The observer, standing at right angles to the line of sight, will see the image inverted.

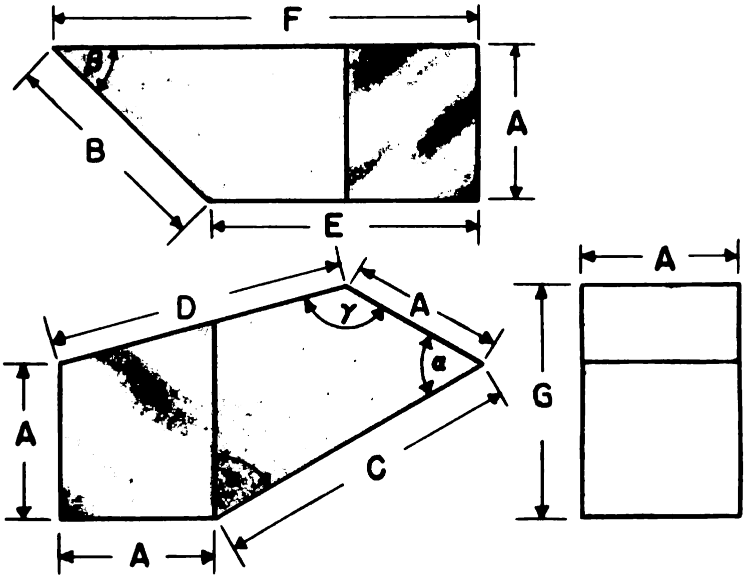


Figure 194. Frankford Arsenal prism No. 5.

$A = 1.00$ $\alpha = 60°$

$n = 1.5170$ $\beta = 45°$ $\gamma = 135°$

$B = 1.4142A = 1.4142$ $C = 2.000A = 2.000$

$D = 1.9318A = 1.9318$ $E = 1.7321A = 1.7321$

$F = 2.7321A = 2.7321$ $G = 1.500A = 1.500$

$d = 2.7437A = 2.7431$ $d/n = 1.8086$

f. Frankford Arsenal Prism No. 6.

This prism is made in one piece. It will deviate the line of sight through an angle of 90° in the horizontal plane and through an angle of 60° in the vertical plane. The prism will invert and revert the image.

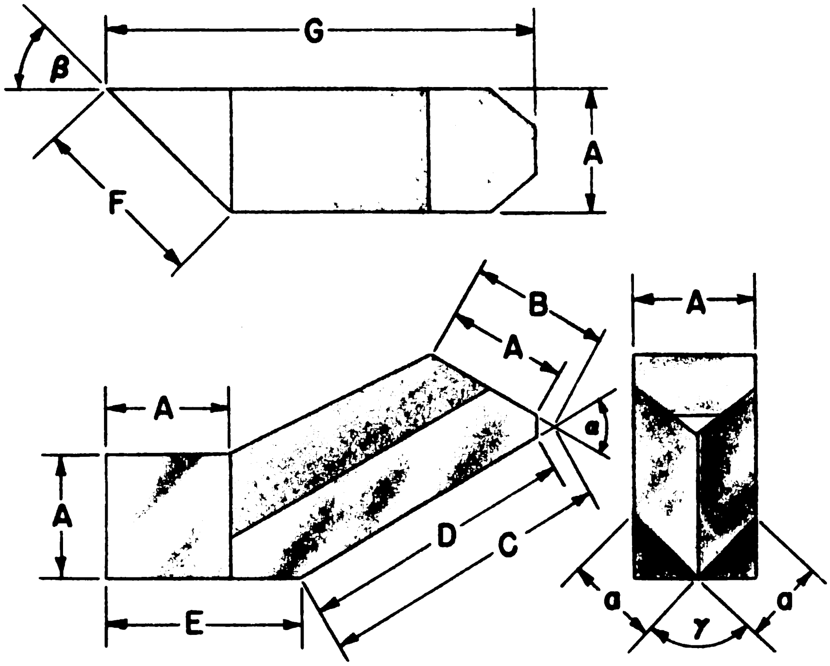


Figure 195. Frankford Arsenal prism No. 6.

$A = 1.00$ $n = 1.5170$ $\alpha = 60^\circ$ $\beta = 45^\circ$ $\gamma = 90^\circ$

$a = 0.7071A = 0.7071$ $B = 1.2071A = 1.2071$

$C = 2.4142A = 2.4142$ $D = 2.2071A = 2.2071$

$E = 1.5774A = 1.5774$ $F = 1.4142A = 1.4142$

$G = 3.4888A = 3.4888$ $H = 1.8107A = 1.8187$

$d = 3.6681A = 3.6681$ $d/n = 2.4180$

g. Frankford Arsenal Prism No. 7.

This prism is made in one piece. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 45° in the vertical plane. The observer, standing at right angles to the line of sight, will see a normal image of the target since the prism neither inverts nor reverts the image.

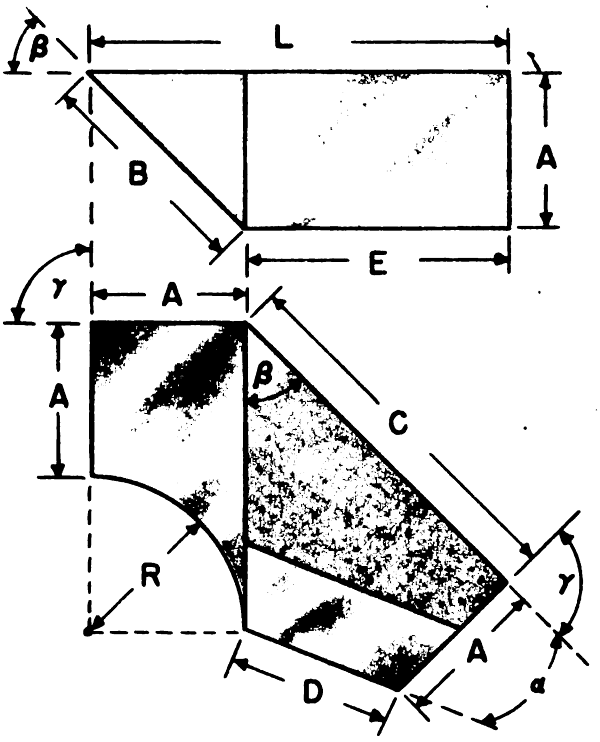


Figure 196. Frankford Arsenal prism No. 7.

$A = 1.00$

$n = 1.5170$

$\alpha = 22°30'$

$\beta = 45°$

$\gamma = 90°$

$B = 1.4142A = 1.4142$

$C = 2.4142A = 2.4142$

$D = 1.0824A = 1.0824$

$E = 1.7071A = 1.7071$

$L = 2.7071A = 2.7071$

$R = A = 1.00$

$d = 4.4142A = 4.4142$

$d/n = 2.9098$

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